The 12-Element Case of Frankl's Conjecture

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Abstract: ***According to Frankl's conjecture, for each union-closed family $\mathcal{F}$ of subsets of finite set X, there exists an element contained in at least half members of $\mathcal{F}$. We give a computer assisted proof that Frankl's conjecture is true if $|X| \leq 12$.***

Index Terms: ***extremal sets, union-closed family, Frankl's conjecture***

1. Introduction

DENOTE by 2^A the family of all subsets of A, and by $[n]$ the set $\{1, \ldots, n\}$. We say that a family $\mathcal{F}$ is *uniform* if all the sets within $\mathcal{F}$ have an equal number of elements. Denote by $\binom{A}{k}$ the uniform family of all subsets of A with k elements. Non-empty collection of sets $\mathcal{F}$ is *union-closed* if for an arbitrary two sets $A, B \in \mathcal{F}$ is also $A \cup B \in \mathcal{F}$. For an arbitrary family $\mathcal{A} \subseteq 2^{[n]}$ let $\overline{\mathcal{A}}$ denote the *closure* of $\mathcal{A}$, the minimum union-closed family in $2^{[n]}$ containing $\mathcal{A}$. Let $\mathcal{F}_\alpha = \{S \in \mathcal{F} \mid \alpha \in S\}$. If $\mathcal{F}$ and $\mathcal{G}$ are any two collections of sets, let $\mathcal{F} \uplus \mathcal{G}$ denote $\{S \cup T \mid S \in \mathcal{F},\ T \in \mathcal{G}\}$.

According to the longstanding Frankl's conjecture (1979), if $\mathcal{F}$ is a union-closed family, then there is an element in $\bigcup \mathcal{F}$ which is contained in at least half elements of $\mathcal{F}$. Following Marković [1] we say that the union-closed family $\mathcal{F}$ is *Frankl's* if it satisfies Frankl's conjecture. Let $n = |\bigcup \mathcal{F}|$. Gao and Yu [11] proved that Frankl's conjecture is satisfied for any union-closed family with $n \leq 8$, Morris [4] proved it for 9 elements, Marković [1] improved the bound to 10, and by now the best obtained result in this manner is by Bošnjak and Marković [2], stating that family is Frankl's if $n \leq 11$. More on the results related to Frankl's conjecture can be found in recent survey by Bruhn and Schaudt [3].

The main tools of our approach are the following definition and lemma (see [1], [2]).

Definition 1. Let $X = \bigcup \mathcal{F}$. A function w that assigns a real nonnegative values to elements of X, such that there exists an element $x \in X$ with $w(x) > 0$, is called *weight function*. The weight of a set $S \subseteq X$ is defined by $w(S) = \sum_{a \in S} w(a)$. The number $t(w) = \frac{1}{2} w(X)$ is the *target weight*.

Lemma 1. *A family $\mathcal{F}$ is Frankl's if and only if there is a weight function $w : X \to \mathbb{R}$, defined on the set $X = \bigcup \mathcal{F}$, such that*

$$\sum_{S \in \mathcal{F}} w(S) \geq t(w)|\mathcal{F}|$$

Manuscript received September 8, 2016. Supported by the Serbian Ministry of Education, Science and Technological Development, project III44006 and by the grant no. 174021.
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It is usually possible to restrict a function w by assigning rational or even integer values to all the elements of family, like we do in this paper. In order to simplify implementation of the previous Lemma, it is convenient to introduce *shares* of elements, sets and families.

Definition 2. Let $\mathcal{F}$ be a union-closed family of sets, and let w be a weight function on $\mathcal{F}$. The share $s(L)$ of the set $L \subseteq \bigcup \mathcal{F}$ is the difference $s(L) = w(L) - t(w)$. The share of an arbitrary family $\mathcal{A} \subseteq \mathcal{F}$ is defined by $s(\mathcal{A}) = \sum_{A \in \mathcal{A}} s(A)$.

By reformulating Lemma 1, we obtain

Corrolary 1. *An arbitrary union-closed family $\mathcal{F}$ is Frankl's if and only if there exists a weight function w, such that $s(\mathcal{F}) \geq 0$.*

Proof. The proof follows from the equality

$$\begin{aligned} s(\mathcal{F}) &= \sum_{S \in \mathcal{F}} s(S) = \sum_{S \in \mathcal{F}} (w(S) - t(w)) \\ &= \sum_{S \in \mathcal{F}} w(S) - t(w)|\mathcal{F}| \end{aligned}$$

□

Denote by S_n the set of all permutation on $[n]$, and for $\phi \in S_n$ set $\phi(A)$ as $\{\phi(x) \mid x \in A\}$. The families $\mathcal{A}, \mathcal{B} \subseteq 2^{[n]}$ are *equivalent*, denoted by $\mathcal{A} \sim \mathcal{B}$, if there exists $\phi \in S_n$, such that $\mathcal{B} = \{\phi(A) \mid A \in \mathcal{A}\}$.

Example 1.

$$\mathcal{A} = \{\{4,5,6\}, \{2,4,6\}, \{3,4,5,6\}\},$$

$$\mathcal{B} = \{\{1,2,3\}, \{1,2,4\}, \{1,2,3,5\}\}$$

For $\phi : \{2,3,4,5,6\} \to [5]$ given by

$$\phi : \begin{pmatrix} 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 1 & 3 & 2 \end{pmatrix},$$

$\mathcal{B}$ is equal to $\{\phi(A) \mid A \in \mathcal{A}\}$, and hence $\mathcal{A} \sim \mathcal{B}$.

2. FC Families

Vaughan [6] introduced the concept of Frankl-complete families, or FC families.

Definition 3. Family $\mathcal{G}$ is an FC family if for any union-closed family $\mathcal{F}$containing subfamily $\mathcal{G}' \sim \mathcal{G}$, there exists an element $x \in \bigcup \mathcal{G}'$ appearing in at least half of the sets of $\mathcal{F}$.

Poonen [5] showed which conditions family $\mathcal{G}$ needs to satisfy to be an FC-family.

The first two claims of the following theorem are proved in [9], the third and fourth in [4], and the last one is from [7].

Theorem 1. *The following families are FC:*

1) *singleton* $\{1\}$
2) *dublet* $\{1,2\}$
3) *an arbitrary three-subset subfamily of* $\binom{[5]}{3}$
4) *an arbitrary four-subset subfamily of* $\binom{[6]}{3}$
5) *an arbitrary four-subset subfamily of* $\binom{[7]}{3}$

Following Bošnjak and Marković [1], [2], for $K \cap S = \emptyset$, let $\mathcal{C}_{K,S} = K \uplus 2^S$ denote the hypercube with the base K and the upper set S. We use the following consequence of Corrolary 1.

Corrolary 2. *Let $\mathcal{F}$ be a union-closed family, and let $S \subseteq \bigcup \mathcal{F}$, $K = \bigcup \mathcal{F} \setminus S$. If there exists a weight function w, such that $\sum_{L \subseteq K} s(\mathcal{C}_{L,S} \cap \mathcal{F}) \geq 0$, then $\mathcal{F}$ is Frankl's.*

Definition 4. The family $\mathcal{G}$ is k-FC family if an arbitrary union-closed family $\mathcal{F} \subseteq 2^{[k]}$ containing $\mathcal{G}' \sim \mathcal{G}$ as a subfamily is Frankl's.

3. Results for $|X| = 12$

Denote by $Q(\mathcal{F})$ and $R(\mathcal{F}, i)$ the following statements:

$$Q(\mathcal{F}): \quad \mathcal{F} \text{ does not contain a family equivalent to some FC family from Theorem 1,} \tag{1}$$

$$R(\mathcal{F}, i): \quad \mathcal{F} \text{ does not contain a family equivalent to some } \mathcal{F}_j,\ j < i, \text{ from Table I.} \tag{2}$$

Definition 5. Let $i \in \{1, \ldots, 33\}$. We say that $\mathcal{F}$ is $\mathcal{F}_i$-*correct* family if it satisfies the following conditions:

1) $\mathcal{F}$ is a union-closed,
2) $\bigcup \mathcal{F} = [12]$,
3) $\mathcal{F}_i \subseteq \mathcal{F}$,
4) $Q(\mathcal{F})$,
5) $R(\mathcal{F}, i)$.

Denote by $S_i = \bigcup \mathcal{F}_i$ and $r_i = |S_i|$. We may assume that $S_i = [r_i]$. The set of hypercubes $\mathcal{C}_{K,S_i} = K \uplus 2^{S_i}$, $K \subseteq [12] \setminus S_i$, partitions the family $2^{[12]}$. Let $\mathcal{C}_{K,S_i}$ be an arbitrary hypercube, corresponding to the base $K \subseteq [12] \setminus S_i$. We say that family $\mathcal{G}$ is $(\mathcal{F}_i, k)$-*correct* if $\mathcal{G} = \mathcal{C}_{K,S_i} \cap \mathcal{F}$, where $k = |K|$ and $\mathcal{F}$ is $\mathcal{F}_i$-correct. It is obvious that, for an $(\mathcal{F}_i, k)$-*correct* family $\mathcal{G}$, the following statements hold:

1) $\mathcal{G}$ is union-closed,
2) $\mathcal{G} \uplus \mathcal{F}_i = \mathcal{G}$,
3) $Q(\mathcal{G})$,
4) $R(\mathcal{G}, i)$.

We say that $\mathcal{G}$ is $(\mathcal{F}_i, k)$-*closed* if it satisfies the first two of the above conditions.
Denote by $\mathbf{1}_A(x)$ the indicator function (equal to 1 if $x \in A$, and equal to 0 otherwise), and by $d_{i,k}^{\,l}$

$$d_{i,k}^{\,l} = \min\{s(\mathcal{G}) \mid \mathcal{G} \text{ is } (\mathcal{F}_i, k)\text{-correct family}, |K| = k, \mathbf{1}_{\mathcal{G}}(K) = l\} \tag{3}$$

Note that the weights of all the elements from K, given in Table I, have the same values. Hence $d_{i,k}^{\,l}$ has the same value for all bases K satisfying $|K| = k$. Instead of considering all the 2^{12-r_i} hypercubes, it is enough to consider only $2(13 - r_i)$ cases, corresponding to $0 \leq |K| \leq 12 - r_i$, and $\mathbf{1}_{\mathcal{F}_i}(K) \in \{0, 1\}$.

Next, we propose a pseudo-code that can be used to find the smallest values of $d_{i,k}^{\,l}$ for every $i \in \{1, \ldots, 33\}$, $k \in \{0, \ldots, 12 - r_i\}$ and $l \in \{0, 1\}$, where the weights of the elements of $\bigcup \mathcal{F}$ assigned by function w are given in the ith row of Table I. We prove later the correctness of the algorithm, and consequently, our main result.

The following simple brute force algorithm for computing $d_{i,k}^{\,l}$ would do the desired calculations.

Algorithm 1.
Input: families $\mathcal{F}_i$, $1 \leq i \leq 33$.
Output: values $d_{i,k}^0$ and $d_{i,k}^1$, for every $i \in \{1, \ldots, 33\}$ and $k \in \{0, \ldots, 12 - r_i\}$.

for all $i \in \{1, \ldots, 33\}$
  for all $k \in \{0, \ldots, 12 - r_i\}$
    set $d_{i,k}^{\,0} \leftarrow \infty$, $d_{i,k}^{\,1} \leftarrow \infty$
    set $K \leftarrow \{r_i + 1, \ldots, r_i + k\}$
    for all *families* $\mathcal{G} \in \{K\} \uplus 2^{[r_i]}$
      if $\mathcal{G}$ *is* $(\mathcal{F}_i, k)$*-correct*
        $d_{i,k}^{\mathbf{1}_{\mathcal{G}}(K)} \leftarrow \min\left\{s(\mathcal{G}), d_{i,k}^{\mathbf{1}_{\mathcal{G}}(K)}\right\}$
return $d_{i,k}^0$ *and* $d_{i,k}^1$

The problem with the above algorithm is that the inner-most loop examines $2^{2^{r_i}}$ families. Even for small values of r_i, for example $r_i = 6$, the number of families we need to check is 2^{64}, which is too much, even for powerful computers. However, it is possible to reduce the number of the cases by considering only $(\mathcal{F}_i, k)$-correct families obtained from a family containing only sets with negative value of share. Denote by $\mathcal{N}_{i,k}$ the following family

$$\mathcal{N}_{i,k} = \{A \in \mathcal{C}_{K,S_i} \mid |K| = k,\ s(A) < 0\} \tag{4}$$
$$= \{N_1, \ldots, N_p\} \tag{5}$$

where $s(N_1) \leq s(N_2) \leq \cdots \leq s(N_p)$. Since $K \subseteq N$ for every $N \in \mathcal{N}_{i,k}$, we have that $N_1 = K$, when $p \geq 1$. Also, since weight of every element shown in Table I is positive, we have that for every $A \subset B$ is $s(A) < s(B)$. Furthermore, for every $i \geq 1$ there does not exist j, $i < j \leq p$ such that $N_j \subset N_i$.
In the case when $\mathcal{N}_{i,k}$ is empty (the only such case is $i = 25$, $k = 9$), the lower bounds are non-negative, and they are easily obtained:

$$d_{i,|K|}^{\,1} = s(\{K\} \uplus \mathcal{F}_i)$$
$$d_{i,|K|}^{\,0} = s(K \cup S_i)$$

Let $|\mathcal{N}| = p$. Subfamilies of $\mathcal{N}$ can be indexed by vectors

$$a = (a_1, \ldots, a_p) \in \{0, 1\}^p \tag{6}$$

Denote by

$$\mathcal{N}_{a,i,k} = \{N_j \mid N_j \in \mathcal{N}_{i,k}, 1 \leq j \leq p, a_j = 1\}. \tag{7}$$

and by $\mathcal{G}_a$ the minimal $(\mathcal{F}_i, k)$-closed family containing $\mathcal{N}_{a,i,k}$, that is

$$\mathcal{G}_a = \overline{\mathcal{N}}_{a,i,k} \uplus \overline{\mathcal{F}_i} \tag{8}$$

TABLE I

12-FC FAMILIES AND THE CORRESPONDING WEIGHT FUNCTIONS USED IN THE PROOF OF LEMMA 4. EACH FAMILY CONTAINS ALSO AN EMPTY SET.

i	$\mathcal{F}_i$	1	2	3	4	5	6	7	$8-12$	$t(w)$
1	$\{1,2,3\}$,$\{1,2,4\}$,$\{1,2,3,5\}$	24	24	18	18	12	2	2	2	55
2	$\{1,2,3\}$,$\{1,4,5,6\}$,$\{2,4,5,6\}$,$\{3,4,5,6\}$	24	24	24	10	10	10	2	2	57
3	$\{1,2,3\}$,$\{1,2,3,4\}$,$\{1,2,3,5\}$,$\{4,5,6\}$	6	6	6	9	9	6	1	1	24
4	$\{1,2,3\}$,$\{1,4,5\}$	11	7	7	7	7	1	1	1	23
5	$\{1,2,3\}$,$\{1,4,5,6\}$,$\{2,4,5,6\}$	6	6	4	4	4	4	1	1	17
6	$\{1,2,3\}$,$\{1,2,3,4\}$,$\{1,2,3,5\}$,$\{1,2,3,6\}$, $\{1,2,3,7\}$	8	8	8	8	8	8	8	2	33
7	$\{1,2,3,4\}$,$\{1,2,3,5\}$,$\{1,2,4,5\}$,$\{1,2,3,6\}$, $\{1,2,4,6\}$,$\{1,2,5,6\}$	5	5	4	4	4	4	1	1	16
8	$\{1,2,3,4\}$,$\{1,2,3,5\}$,$\{1,2,3,6\}$,$\{1,2,3,7\}$	8	8	8	6	6	6	6	2	29
9	$\{1,2,3\}$,$\{1,2,3,4\}$,$\{1,2,3,5\}$,$\{1,2,3,6\}$	10	10	10	8	8	8	2	2	33
10	$\{1,2,3\}$,$\{1,2,3,4\}$,$\{1,2,3,5\}$,$\{4,6,7\}$	3	3	3	3	3	3	3	1	13
11	$\{1,2,3\}$,$\{1,2,3,4\}$,$\{4,5,6\}$	8	8	8	14	6	6	2	2	31
12	$\{1,2,3,4\}$,$\{1,2,3,5\}$,$\{1,2,3,6\}$	10	10	10	8	8	8	2	2	33
13	$\{1,2,3\}$,$\{1,2,4,5\}$,$\{1,3,4,5\}$	12	12	12	8	8	2	2	2	33
14	$\{1,2,3\}$,$\{1,4,5,6,7\}$,$\{2,4,5,6,7\}$,$\{3,4,5,6,7\}$	3	3	3	3	3	3	3	1	13
15	$\{1,2,3\}$,$\{1,4,5,6\}$,$\{1,2,4,5,6\}$	7	7	4	4	4	4	1	1	18
16	$\{1,2,3,4\}$,$\{1,2,3,5\}$,$\{1,2,4,5\}$	9	9	8	8	8	2	2	2	28
17	$\{1,2,3\}$,$\{4,5,6\}$	3	3	3	3	3	3	1	1	12
18	$\{1,2,3\}$,$\{1,2,4\}$	4	4	4	4	1	1	1	1	12
19	$\{1,2,3\}$,$\{1,2,4,5,6\}$,$\{1,3,4,5,6\}$	8	8	8	6	6	6	2	2	27
20	$\{1,2,3\}$,$\{1,4,5,6\}$	6	6	6	4	4	4	2	2	21
21	$\{1,2,3\}$,$\{1,2,3,4\}$,$\{1,2,3,5\}$	3	3	3	3	3	1	1	1	11
22	$\{1,2,3\}$,$\{1,2,4,5\}$	10	10	8	6	6	2	2	2	27
23	$\{1,2,3\}$,$\{1,2,4,5,6\}$	4	4	4	2	2	2	1	1	12
24	$\{1,2,3\}$,$\{4,5,6,7\}$	3	3	3	2	2	2	2	1	11
25	$\{1,2,3\}$	3	3	3	1	1	1	1	1	9
26	$\{1,2,3,4\}$,$\{1,2,3,5\}$,$\{1,2,3,4,6\}$	6	6	6	6	6	4	2	2	23
27	$\{1,2,3,4\}$,$\{1,2,3,4,5\}$,$\{1,2,3,4,6\}$	2	2	2	2	2	2	1	1	9
28	$\{1,2,3,4\}$,$\{1,2,3,5\}$	3	3	3	2	2	1	1	1	10
29	$\{1,2,3,4,5\}$,$\{1,2,3,4,6\}$	3	3	3	3	3	3	1	1	12
30	$\{1,2,3,4\}$,$\{1,2,3,4,5\}$	3	3	3	3	3	1	1	1	11
31	$\{1,2,3,4\}$,$\{1,2,5,6\}$	3	3	3	3	3	3	1	1	12
32	$\{1,2,3,4\}$	5	5	5	5	2	2	2	2	18
33	$\{1,2,3,4,5\}$	4	4	4	4	4	2	2	2	17

Hence, the more efficient version of the algorithm would examine only $(\mathcal{F}_i, k)$-closed families $\mathcal{G}_a$.

Algorithm 2.
Input: families $\mathcal{F}_i$, $1 \leq i \leq 33$.
Output: values $d^0_{i,k}$ *and* $d^1_{i,k}$, *for every* $i \in \{1, \ldots, 33\}$ *and* $k \in \{0, \ldots, 12 - r_i\}$.

for all $i \in \{1, \ldots, 33\}$ **do**
  for all $k \in \{0, \ldots, 12 - r_i\}$ **do**
    set $d^0_{i,k} \leftarrow \infty$, $d^1_{i,k} \leftarrow \infty$
    set $K \leftarrow \{r_i + 1, \ldots, r_i + k\}$
    for all $a \in \{0,1\}^p$ **do**
      set $\mathcal{G}_a \leftarrow \overline{\mathcal{N}}_{a,i,k} \uplus \overline{\mathcal{F}_i}$
      if $Q(\mathcal{G}_a)$ ***and*** $R(\mathcal{G}_a, i)$ **then**
        $d^{\mathbf{1}_{\mathcal{G}_a}(K)}_{i,k} \leftarrow \min\left\{s(\mathcal{G}_a), d^{\mathbf{1}_{\mathcal{G}_a}(K)}_{i,k}\right\}$
  return $d^0_{i,k}$ *and* $d^1_{i,k}$

Even this algorithm, though more efficient than the previous one, is to demanding. It is possible to further reduce the number of calculations with backtracking algorithm. We now introduce some notation that makes it easier to explain the steps of the backtracking algorithm.
Let a be a vector given by (6) and

$$b = (a_1, \ldots, a_t),\ 1 \leq t \leq p \tag{9}$$

Denote by $v(b)$:

$$v(b) = s(\mathcal{G}_b) + \sum_{\substack{t < i \leq p \\ N_i \notin \mathcal{G}_b}} s(N_i). \tag{10}$$

Let a and b be vectors given by (6) and (9), respectively. We now present the backtracking algorithm, that discards large quantity of families that have a share greater than $d^l_{i,k}$.

Algorithm 3 (backtracking()).
Input: family $\mathcal{F}_i$, *family* $\mathcal{N}_{i,k}$, *integer* k, *vector* $b = (a_1, \ldots, a_t)$

  set $\mathcal{G}_b \leftarrow \overline{\mathcal{N}_b} \uplus \overline{\mathcal{F}_i}$
  if $Q(\mathcal{G}_b)$ ***and*** $R(\mathcal{G}_b, i)$ ***then***
    if $v(b) < d^{\mathbf{1}_{\mathcal{G}_b}(K)}_{i,k}$ ***then***
      set $d^{\mathbf{1}_{\mathcal{G}_b}(K)}_{i,k} \leftarrow \min\left\{s(\mathcal{G}_b), d^{\mathbf{1}_{\mathcal{G}_b}(K)}_{i,k}\right\}$
      if $t < |\mathcal{N}_{i,k}|$ **then**
        backtracking($\mathcal{F}_i$, $\mathcal{N}_{i,k}$, k, $(a_1, \ldots, a_t, 1)$*)*
        if $N_{t+1} \notin \mathcal{G}_b$ **then**
          backtracking($\mathcal{F}_i$, $\mathcal{N}_{i,k}$, k, $(a_1, \ldots, a_t, 0)$*)*

The following algorithm can be used to call backtracking algorithm for each $i \in \{1, \ldots, 33\}$, thus calculating the values $d^l_{i,k}$.

Algorithm 4.
Input: families $\mathcal{F}_i$, $1 \le i \le 33$.
Output: values $d^0_{i,k}$ *and* $d^1_{i,k}$*, for every* $i \in \{1, 2, \ldots, 33\}$ *and* $k \in \{0, 1, \ldots, 12 - r_i\}$.

for all $i \in \{1, \ldots, 33\}$ **do**
  for all $k \in \{0, \ldots, 12 - r_i\}$ **do**
    set $d^{\,0}_{i,k} \leftarrow \infty$, $d^{\,1}_{i,k} \leftarrow \infty$
    set $K \leftarrow \{r_i + 1, \ldots, r_i + k\}$
    calculate $\mathcal{N}_{i,k}$
    backtracking($\mathcal{F}_i$, $\mathcal{N}_{i,k}$, k, $b = (1)$*)*
    if $k > 0$ **then**
      backtracking($\mathcal{F}_i$, $\mathcal{N}_{i,k}$, k, $b = (0)$*)*
  return $d^0_{i,k}$ *and* $d^1_{i,k}$

Lemma 2. *Using the Algorithm 4 we obtain the values of* $d^0_{i,k}$ *and* $d^1_{i,k}$ *for every* $i \in \{1, 2, \ldots 33\}$ *and* $k \in \{0, 1, \ldots 12 - r_i\}$.

Proof. Let a and b be vectors given by (6) and (9). Let $\mathcal{G}_a$ and $\mathcal{G}_b$ be families given by (8), obtained from the vectors a and b, respectively. We have the following simple observations

1) $\mathcal{G}_b \subseteq \mathcal{G}_a$,
2) if $Q(\mathcal{G}_a)$ then $Q(\mathcal{G}_b)$,
3) if $R(\mathcal{G}_a, i)$ then $R(\mathcal{G}_b, i)$,
4) $v(b) \le s(\mathcal{G}_a)$, thus if $v(b) \ge d^{\,l}_{i,k}$ then $s(\mathcal{G}_a) \ge d^{\,l}_{i,k}$.

Hence, when $\mathcal{G}_b$ contains as a subfamily some FC-family or $\mathcal{F}_j$, for some $j < i$, or when $v(b) \ge d^{\,l}_{i,k}$, then all the vectors a, having b as a prefix, can be skipped. Let $i \in \{1, \ldots, 33\}$ and let $\mathcal{F}$ contain $\mathcal{F}_i$, as a subfamily. Let $\mathcal{G}$ be an $(\mathcal{F}_i, k)$-correct family such that $s(\mathcal{G}) = d^{\mathbf{1}_{\mathcal{G}}(K)}_{i,k}$, where $k \in \{0, \ldots, r_i\}$. Let $a \in \{0, 1\}^p$ be the vector corresponding to the sets from $\mathcal{N}_{i,k}$ that are included in $\mathcal{G}$. We prove that the value of $d^{\mathbf{1}_{\mathcal{G}}(K)}_{i,k}$ can be obtained by the backtracking algorithm. Let $l = |\mathcal{N}_{a,i,k}|$.
We proceed by induction on l. Obviously, for $l = 0$ the value of $d^{\mathbf{1}_{\mathcal{G}}(K)}_{i,k}$ is obtained. Thus, we may assume that $l \ge 1$ and that for every $0 \le j < l$, the family induced by the vector $(a_1, \ldots, a_j)$ with $d^{\mathbf{1}_{\mathcal{G}}(K)}_{i,k}$ can be reached by the backtracking algorithm. Let $\mathcal{H} = \mathcal{G} \cap \mathcal{N}_{i,k}$, and let m be the largest number, such that $N_m \in \mathcal{H}$, $1 \le m \le l$ and $\overline{\mathcal{G} - N_m} \ne \mathcal{G}$. Furthermore, let n be the largest number, such that $1 \le n < m$, $N_n \in \mathcal{G}$, and in case such N_n does not exists, let $n = 0$. By the induction hypothesis, family $\mathcal{G}_b = \overline{\mathcal{N}}_{b,i,k} \uplus \overline{\mathcal{F}_i}$, where $b = (a_1, \ldots, a_n)$, can be reached. Since value of $v(b)$ from (10) is smaller or equal to $d^{\mathbf{1}_{\mathcal{G}}(K)}_{i,k}$ the backtracking algorithm will continue with recursive calls, all the way to the $a = (a_1, \ldots, a_m)$, thus the values of $d^l_{i,k}$ for $l = 0, 1$ are obtained. □

Example 2. Let $i = 32$, then $\mathcal{F}_{32} = \{\{1, 2, 3, 4\}\}$, $S_i = \bigcup \mathcal{F}_{32} = [4]$, $r_i = |S_i| = 4$. The weights are $w(1) = w(2) = w(3) = w(4) = 5$, and $w(x) = 2$ for all $x > 4$; $t(w) = 18$. When $k = 0$, then the base of the hypercube is $K = \emptyset$, and in case when $k = 1$ the base of the hypercube is $K = \{5\}$. In both cases sets with negative share are

$$\mathcal{N} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,3\}, \{2,3\}, \{1,2\}, \{1,4\}, \{2,4\}, \{3,4\}, \{1,2,4\}, \{1,2,3\}, \{1,3,4\}, \{2,3,4\}\}.$$

The values $d^{\,1}_{32,0} = -16$ and $d^{\,0}_{32,1} = 4$ are calculated after the sequence of recursive calls listed in Tables II and III, respectively.

TABLE III
THE SEQUENCE OF RECURSIVE CALLS, $k = 1$, EXAMPLE 2.

a	$\mathcal{N}_{a,i,k}$	backtracking?
$K \in \mathcal{G}_a$		
1	$\{\{5\}\}$	yes, $\mathcal{G}_a$ contains FC family $\{\{5\}\}$
$K \notin \mathcal{G}_a$		
01	$\{\{1,5\}\}$	yes, $\mathcal{G}_a$ contains FC family $\{\{1,5\}\}$

The values of $d^{\,l}_{i,k}$, given in the Table IV, are obtained with program written in Java programming language implementing the Algorithm 4. It takes about five minutes to calculate all these values, on 64-bit Acer laptop with Intel's i7 processor, on 2.4 GHz, with 16 GB of RAM. The largest number of recursive calls for some of the cases is 34437982 for $(i, k) = (14, 2)$. Note that values of (i, k, l) are left out from the Table IV when:

- $1 \le i \le 33$, $k = 0$, $l = 0$.
  This is done because an empty set is implicitly included in all of the considered families;
- $1 \le i \le 33$, $k \in \{1, 2\}$, $l = 1$.
  According to Theorem 1, any family containing a singleton or doublet is Frankl's.
- $i \ge 18$, $k = 3$, $l = 1$.
  $\mathcal{G}_a$ contains a family equivalent to $\mathcal{F}_{17} = \{\{1,2,3\},\{4,5,6\}\}$, the first 3-set is K, and the second one is an element of $\mathcal{F}_j$, $18 \le j \le 24$.
- $i \ge 25$, $k = 3$, $l = 1$.
  $\mathcal{G}_a \supseteq K \sim \mathcal{F}_{25} = \{\{1,2,3\}\}$.
- $i = 25$, $k = 4$, $l = 1$.
  $\mathcal{G}_a \supseteq \mathcal{F}_{25} \cup \{K\} = \{\{1,2,3\},\{4,5,6,7\}\} \sim \mathcal{F}_{24}$.
- $i = 33$, $k = 4$, $l = 1$.
  $\mathcal{G}_a \supseteq K = \{\{1,2,3,4\}\} \sim \mathcal{F}_{32}$.

Lemma 3. *Suppose* A*,* B *and* C *are three different three-member sets. Then at least one of the following three statements is true:*

1) *The union of some two of the sets* A*,* B *and* C *is a five-element set.*
2) *There are two pairs of disjoint sets among* A*,* B*,* C*.*
3) *The family* $\{A, B, C\}$ *is FC.*

Proof. Since the size of the intersection of two different three-member sets is 0, 1 or 2, we have the following possibilities:

1) The intersection of some two of the sets is one-element set. Then the size of their union is five, and the first statement is true.
2) Neither of the two-set intersections is one-element set.
   a) Some two of the three sets A, B, C are disjoint. Assume that $A \cap B = \emptyset$. Then $|C \cap A| = 0$ or $|C \cap B| = 0$ (otherwise, it would be $|C \cap A| = |C \cap B| = 2$, implying $|C| \ge 4$), and the second statement is true.

b) Intersection of any two of the sets A, B and C is a nonempty set. Then $|A \cap B| = |A \cap C| = |B \cap C| = 2$. Let $|A \cap B \cap C| = q$. The case $q = 0$ is impossible. If $q = 1$, then $|A \cup B \cup C| = 4$, and if $q = 2$, then $|A \cup B \cup C| = 5$. In both cases the family $\{A, B, C\}$ is FC by Theorem 1, hence the third statement is true.

□

Lemma 4. *The families $\mathcal{F}_i$ listed in Table I are 12-FC, $1 \leq i \leq 33$.*

Proof. Let the values of $d^l_{i,k}$, $0 \leq k \leq 12 - r_i$, $l \in \{0, 1\}$, given in the Table IV, be the smallest values of share of all the families that contain family equivalent to $\mathcal{F}_i$ as a subset, and do not contain as a subset a family equivalent to some FC family from Theorem 1 or $\mathcal{F}_j$, for any $1 \leq j < i$. Let $\mathcal{K}_k = \{K | K \subseteq X \setminus S_i, |K| = k\}$. We have that $|\mathcal{K}_k| = \binom{12-r_i}{k}$. $\mathcal{C}_{K,S_i} \cap \mathcal{F} \neq \emptyset$ for $|K| = 0$ and $|K| = 12 - r_i$, since $S_i \in \mathcal{F}$ and $X = [12] \in \mathcal{F}$, and in both cases $|\mathcal{K}_k| = 1$. Let W_i denote smaller of the two values of the share bounds (when $\mathcal{F}$ contains, and when it does not contain a set K) of the upper hypercube, that is $W_i = \min\{d^{\,0}_{i,12-r_i}, d^{\,1}_{i,12-r_i}\}$.

Case $i = 7, 8$ or $17 \leq i \leq 33$:

From Table I could be seen that $d^{\,l}_{i,k} < 0$, and hence $s(\mathcal{C}_{K,S_i} \cap \mathcal{F}) < 0$ is possible only when $k = 0$. But, for all such i, the negative share of the lowest hypercube is compensated by the share of the uppermost hypercube: $s(\mathcal{F}) \geq d^{\,1}_{i,0} + W_i \geq 0$.

Case $1 \leq i \leq 6$ or $9 \leq i \leq 16$:

$d^{\,l}_{i,k} < 0$ is possible only if $k \in \{0, 3\}$. We have $12 - r_i \in \{5, 6, 7\}$. Let

$$m_i = \begin{cases} 3, & \text{if } 12 - r_i = 5 \\ 4, & \text{if } 12 - r_i \in \{6, 7\}. \end{cases}$$

If $|\mathcal{K}_3| \geq m_i$, then the family $\mathcal{K}_3 \subset \mathcal{F}$ is FC by Theorem 1. Otherwise, if $|\mathcal{K}_3| \leq m_i - 2$, then $s(\mathcal{F}) \geq d_{i,0} + W_i + (m_i - 2)d^1_{i,3} \geq 0$ (see Table V). In the remaining case $|\mathcal{K}_3| = m_i - 1$ the inequality $s(\mathcal{F}) \geq 0$ is also true; indeed, from Lemma 3 and Table V it follows that:

- $12 - r_i = 7$: $s(\mathcal{F}) \geq d^{\,1}_{i,0} + (m_i - 1)d^{\,1}_{i,3} + \min\{d^{\,1}_{i,5}, 2d^{\,1}_{i,6}\} + W_i \geq 0$
- $12 - r_i = 6$: $s(\mathcal{F}) \geq d^{\,1}_{i,0} + (m_i - 1)d^{\,1}_{i,3} + d^{\,1}_{i,5} + d^{\,1}_{i,6} \geq 0$
- $12 - r_i = 5$: $s(\mathcal{F}) \geq d^{\,1}_{i,0} + (m_i - 1)d^{\,1}_{i,3} + d^{\,1}_{i,5} \geq 0$

□

Theorem 2. *If $\mathcal{F}$ is a union-closed family, $X = \bigcup \mathcal{F}$ and $|X| = 12$, then $\mathcal{F}$ is Frankl's.*

Proof. Suppose $\mathcal{F}$ is not Frankl's. Then $\mathcal{F}$ does not contain a singleton, nor does $\mathcal{F}$ contain a doublet. From Lemma 4 it follows that $\mathcal{F}$ does not contain

- a subfamily equivalent to $\mathcal{F}_{25} = [3]$,
- a subfamily equivalent to $\mathcal{F}_{32} = [4]$,
- a subfamily equivalent to $\mathcal{F}_{33} = [5]$.

TABLE V
CALCULATIONS ACCOMPANYING THE PROOF OF LEMMA 4. $b \in \{0, 1\}$, i IS THE NUMBER OF ROW IN TABLE IV WITH $d^1_{i,3} < 0$, $m_i \in \{3, 4\}$, $r_i = |\bigcup \mathcal{F}_i|$.

$\lvert K \rvert$	**0**	**3**	**5**	**6**	**7**				
$i \setminus b$	1	1	1	1	1	0	m_i	r_i	$s(\mathcal{F}) \geq$
1	60	-36	30	58	86	55	4	5	37
2	30	-40	56	96			4	6	62
3	6	-15	15	30			4	6	6
4	-3	-6	6	12	18	23	4	5	3
5	2	-2	20	29			4	6	45
6	60	-18	140				3	7	164
9	30	-23	56	96			4	6	113
10	-9	-4	24				3	7	7
11	-17	-5	26	38			4	6	32
12	27	-7	49	87			4	6	142
13	3	-1	23	45	64	33	4	5	56
14	6	-9	32				3	7	20
15	3	-5	14	24			4	6	26
16	-2	-10	24	48	64	28	4	5	20

Thus, all sets in $\mathcal{F}$ (except the set $\emptyset$) have 6 or more elements. Let the weight function w be such that $w(x) = 1$ for all $x \in X$. Then $t(w) = 6$, $s(\emptyset) + s(X) = 0$, and for all non empty sets $A \in \mathcal{F}$ we have $s(A) \geq 0$. Therefore, $s(\mathcal{F}) \geq 0$, and $\mathcal{F}$ is Frankl's, which is a contradiction, thus proving the theorem statement. □

If there exists a counterexample $\mathcal{F}$ to Frankl's conjecture, when $\bigcup \mathcal{F} < 12$, then it would be possible to construct (see [2]) a counterexample $\mathcal{F}'$, when $\bigcup \mathcal{F}' = 12$. Therefore, if $\mathcal{F}$ is a union-closed family, $X = \bigcup \mathcal{F}$ and $|X| \leq 12$, then $\mathcal{F}$ is Frankl's.

Lo Faro [8] and later Roberts and Simpson [10] proved that if minimal counterexample, in terms of $|\bigcup \mathcal{F}|$, has m elements than any counterexample has at least $4m - 1$ sets. As a direct consequence of this statement we have the following corollary.

Corrolary 3. *Frankl's conjecture is satisfied for any union closed family with at most* 50 *sets.*

4. Conclusion

Implementing technique presented in this paper on families having 13 or more elements is probably a difficult task. The main problem is that, when $\bigcup \mathcal{F}$ has only one more element, the number of families that should be considered grows exponentially. That, even with very efficient algorithm, demands too many calculations.

5. Acknowledgement

We thank the referees for many useful suggestions that helped us improve the article.

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TABLE II
THE SEQUENCE OF RECURSIVE CALLS, EXAMPLE 2.

a	$\mathcal{N}_a$	backtracking?
$K \in \mathcal{G}_a$		
1	$\{\emptyset\}$	no
11	$\{\emptyset, \{1\}\}$	yes, $\mathcal{G}_a$ contains FC family $\{\{1\}\}$
101	$\{\emptyset, \{2\}\}$	yes, $\mathcal{G}_a$ contains FC family $\{\{2\}\}$
1001	$\{\emptyset, \{3\}\}$	yes, $\mathcal{G}_a$ contains FC family $\{\{3\}\}$
10001	$\{\emptyset, \{4\}\}$	yes, $\mathcal{G}_a$ contains FC family $\{\{4\}\}$
100001	$\{\emptyset, \{1,3\}\}$	yes, $\mathcal{G}_a$ contains FC family $\{\{1,3\}\}$
1000001	$\{\emptyset, \{2,3\}\}$	yes, $\mathcal{G}_a$ contains FC family $\{\{2,3\}\}$
10000001	$\{\emptyset, \{1,2\}\}$	yes, $\mathcal{G}_a$ contains FC family $\{\{1,2\}\}$
100000001	$\{\emptyset, \{1,4\}\}$	yes, $\mathcal{G}_a$ contains FC family $\{\{1,4\}\}$
1000000001	$\{\emptyset, \{2,4\}\}$	yes, $\mathcal{G}_a$ contains FC family $\{\{2,4\}\}$
10000000001	$\{\emptyset, \{3,4\}\}$	yes, $\mathcal{G}_a$ contains FC family $\{\{3,4\}\}$
100000000001	$\{\emptyset, \{1,2,4\}\}$	yes, $\mathcal{G}_a$ contains $\{\{1,2,4\}\}$ equivalent to $\mathcal{F}_{25}$
1000000000001	$\{\emptyset, \{1,2,3\}\}$	yes, $\mathcal{G}_a$ contains $\{\{1,2,3\}\}$ equivalent to $\mathcal{F}_{25}$
10000000000001	$\{\emptyset, \{1,3,4\}\}$	yes, $\mathcal{G}_a$ contains $\{\{1,3,4\}\}$ equivalent to $\mathcal{F}_{25}$
100000000000001	$\{\emptyset, \{2,3,4\}\}$	yes, $\mathcal{G}_a$ contains $\{\{2,3,4\}\}$ equivalent to $\mathcal{F}_{25}$
100000000000000	$\{\emptyset\}$	no, $t = \lvert\mathcal{N}\rvert$, $d^1_{33,0} = s(\mathcal{G}_a) = -16$

TABLE IV
LOWER BOUNDS $d^l_{i,k}$ ON HYPERCUBE SHARES IN LEMMA 4, $1 \le i \le 33$, $0 \le k \le 12 - \lvert\bigcup \mathcal{F}_i\rvert$, $l \in \{0,1\}$.

$\lvert K\rvert$	**0**	**1**	**2**	**3**		**4**		**5**		**6**		**7**		**8**		**9**			
$i \setminus l$	1	0	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	$12 - r_i$	r_i
1	60	38	15	−36	13	0	43	30	51	58	53	86	55					7	5
2	30	47	45	−40	11	8	53	56	55	96	57							6	6
3	6	19	12	−15	6	0	20	15	23	30	24							6	6
4	−3	12	3	−6	1	0	19	6	21	12	22	18	23					7	5
5	2	12	9	−2	4	5	15	20	16	29	17							6	6
6	60	25	1	−18	9	52	31	140	33									5	7
7	22	11	8	7	4	26	14	53	15	80	16							6	6
8	58	18	7	15	6	77	27	155	29									5	7
9	30	14	3	−23	1	8	28	56	31	96	33							6	6
10	−9	6	0	−4	1	9	12	24	13									5	7
11	−17	8	16	−5	9	8	27	26	29	38	31							6	6
12	27	14	12	−7	1	3	28	49	31	87	33							6	6
13	3	19	18	−1	20	1	26	23	29	45	31	64	33					7	5
14	6	3	6	−9	1	9	12	32	13									5	7
15	3	8	11	−5	7	0	13	14	17	24	18							6	6
16	−2	10	4	−10	12	0	20	24	24	48	26	64	28					7	5
17	−12	2	6	0	0	0	8	8	11	12	12							6	6
18	−8	4	4		1	0	8	6	9	14	10	19	11	24	12			8	4
19	−1	16	13		14	0	19	19	25	41	27							6	6
20	−18	2	1		0	3	16	17	19	27	21							6	6
21	−7	5	3		0	0	7	9	9	20	10	27	11					7	5
22	−8	15	12		2	1	17	14	23	31	25	43	27					7	5
23	−4	3	4		2	2	10	10	11	18	12							6	6
24	−10	4	3		0	3	10	10	11									5	7
25	−9	1	2		3		1	0	4	3	6	5	7	7	8	9	9	9	3
26	0	8	5		2	8	19	43	21	65	23							6	6
27	−7	0	0		0	1	6	16	8	24	9							6	6
28	−5	1	0		1	4	3	8	7	15	9	22	10					7	5
29	0	5	3		6	6	0	7	10	18	12							6	6
30	−6	3	3		0	1	3	3	6	9	8	14	10					7	5
31	−6	7	2		6	6	10	5	11	12	12							6	6
32	−16	4	6		1	0	5	0	8	7	13	12	16	16	18			8	4
33	−14	5	5		6		5	1	8	9	14	14	17					7	5