Row Space Cardinalities Above $2^{n-2} + 2^{n-3}$

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Abstract: ***Denote by $\mathcal{B}_{n,m}$ the set of $n \times m$ Boolean matrices, and let $\mathcal{B}_n = \mathcal{B}_{n,n}$. Denote by $\mathrm{R}(A)$ the row space of $A \in \mathcal{B}_{n,m}$, i.e. the subspace of $\{0,1\}^m$ spanned by the rows of A, with Boolean operations and, or. Denote $\mathcal{R}_n = \{a \mid a = |\mathrm{R}(A)|,\ A \in \mathcal{B}_n\}$. Konieczny determined $\mathcal{R}_n \cap [2^{n-1}, 2^n]$, and other authors determined some information about $\mathcal{R}_n \cap [2^{n-2} + 2^{n-3}, 2^{n-1})$. Here the set $\mathcal{R}_n \cap [2^{n-2} + 2^{n-3}, 2^{n-1})$ is completely determined.***

Index Terms: *row space cardinality*

1. Introduction

DENOTE by $\mathcal{B}_{n,m}$ the set of $n \times m$ Boolean matrices, and let $\mathcal{B}_n = \mathcal{B}_{n,n}$. Denote by $\mathrm{R}(A)$ the row space of $A \in \mathcal{B}_{n,m}$, i.e. the subspace of $\{0,1\}^m$ generated by the rows of A, with Boolean operations and, or. Analogously, let $\mathrm{C}(A)$ denote the column space A; it is known [1] that $|\mathrm{C}(A)| = |\mathrm{R}(A)|$. Denote $\mathcal{R}_n = \{a \mid a = |\mathrm{R}(A)|,\ A \in \mathcal{B}_n\}$. Obviously, $\mathcal{R}_n \subseteq [1, 2^n]$. Konieczny proved [2] that $\mathcal{R}_n \cap (2^{n-1}, 2^n] = \{2^{n-1} + 2^k \mid 0 \le k \le n-1\}$, and conjectured that $[1, 2^{n-1}] \subset \mathcal{R}_n$. Li and Zhang [3] disproved the conjecture, by showing that $2^{n-1} - 1 \notin \mathcal{R}_n$ if $n > 6$. Hong [4] proved that $\mathcal{R}_n \cap (2^{n-1} - 4, 2^{n-1}) = \emptyset$ if $n \ge 8$. Yu [5] proved that

$$\mathcal{R}_n \cap [2^{n-1} - n + 6, 2^{n-1} - 1] = \emptyset, \quad n \ge 7$$

Furthermore, Hong [6] proved that

$$\mathcal{R}_n \cap ((2^{n-1} - 2^{n-5}, 2^{n-1} - 2^{n-6}) \cup (2^{n-1} - 2^{n-6}, 2^{n-1})) = \emptyset, \quad n \ge 7$$

i.e. that the set $\mathcal{R}_n^0 = \mathcal{R}_n \cap [1, 2^{n-1}]$ has at least two continuous gaps. He also proved that $2^{n-1} - 2^{n-5},\ 2^{n-1} - 2^{n-6} \in \mathcal{R}_n$. Zhang, Hong and Kan [7] investigated the matrices $A \in \mathcal{B}_{n-1,n}$ and proved that for such a matrices, if $|R(A)| \in (2^{n-2} + 2^{n-3}, 2^{n-1}]$, then $|R(A)| = 2^{n-2} + 2^{n-3} + 2^s$, $0 \le s \le n-3$. Breen [8] verified that $\mathcal{R}_7^0 = [1, 64] \setminus \{61, 63\}$ and obtained

$$\mathcal{R}_8^0 = [1, 128] \setminus \{109, 111, 117, 119, 121, 122, 123, 125, 126, 127\}.$$

Živković [9] obtained $\mathcal{R}_9$,

$$\begin{aligned}\mathcal{R}_9^0 \quad = \quad & [1, 190] \cup [192, 204] \cup \{206\} \cup [208, 212] \\ & \cup \{214, 216, 220\} \cup [224, 228] \\ & \cup \{230, 232, 236, 240, 248, 256\}.\end{aligned}$$

Let

$$\begin{aligned}
\mathcal{A}_{n,3} &= \{2^{n-2} + 2^{n-3} + 2^i \mid 0 \le i \le n-4, n \ge 4\} \\
\mathcal{A}_{n,4} &= \{2^{n-2} + 2^{n-3} + 2^i + 2^j \mid \\
&\quad 0 \le j < i \le n-4, n \ge 5\} \\
\mathcal{A}'_{n,5} &= \{2^{n-2} + 2^{n-3} + 2^i + 2^{k+1} + 2^k \mid \\
&\quad 0 \le k,\ k+2 \le i \le n-4, n \ge 6\} \\
\mathcal{A}''_{n,5} &= \{2^{n-2} + 2^{n-3} + 2^i + 2^j + 2^k \mid 1 \le k, \\
&\quad k+2 \le j < i,\ i+j \le n+k-5, n \ge 11\} \\
\mathcal{A}_n &= (\mathcal{A}_{n,3} \cup \mathcal{A}_{n,4} \cup \mathcal{A}'_{n,5} \cup \mathcal{A}''_{n,5}) \qquad (1)
\end{aligned}$$

The conjecture $\mathcal{R}_n \cap (2^{n-2} + 2^{n-3}, 2^{n-1}] = \mathcal{A}_n$ [9], can easily be verified for $n \le 9$, and it is proved here for general case, see Theorem 6.

We introduce now some notation. Denote $\mathrm{r}(A) = |\mathrm{R}(A)|$, $A \in \mathcal{B}_{n,m}$. For a square matrix $A \in \mathcal{B}_n$ define the **density** of A by $\rho(A) = \mathrm{r}(A)/2^n$ of A. Denote by I_n and 0_n, $n \ge 1$, the $n \times n$ unit matrix and the $n \times n$ zero matrix, respectively. Then $\rho(I_n) = 1$, $\rho(0_n) = 2^{-n}$ and $2^{-n} \le \rho(A) \le 1$ for all $A \in \mathcal{B}_n$, because $1 \le \mathrm{r}(A_n) \le 2^n$. Denote by $\mathbf{0}_{n,m}$, $\mathbf{1}_{n,m}$ the matrices from $\mathcal{B}_{n,m}$ whose all elements are 0, 1, respectively.

Let $A, B \in \mathcal{B}_{n,m}$. We say that A is **equivalent** to B and write $A \equiv B$, if there exist two permutation matrices P and Q such that $A = PBQ$. We say that $A \le B$ holds (A is lexicographically less than or equal to B) if the row obtained by concatenating all rows of A is lexicographically less or equal than the row obtained by concatenating all rows of B. Denote by $\mathrm{can}(A)$ the **canonical** form of $A \in \mathcal{B}_{n,m}$, the lexicographically smallest matrix among all the matrices equivalent to A.

Let $A \in \mathcal{B}_{n,m}$. Denote by $A(i,j)$ the (i,j) element of A. Let $A(i,*)$ and $A(*,j)$ denote the ith row and the jth column of A, respectively. If p and q are a positive integers, let A^{-p*} (A^{-*p}) denote the matrix $C \in \mathcal{B}_{n-1,m}$ ($D \in \mathcal{B}_{n,m-1}$) obtained by excluding pth row (column) from A, and let $A^{-p*,-q*}$ ($A^{-*p,-q*}$) denote the matrix $C \in \mathcal{B}_{n-2,m}$ ($D \in \mathcal{B}_{n,m-2}$) obtained by excluding pth and qth row (column) from A. Let $A \in \mathcal{B}_{n,m}$ and let p be a positive integer. Denote

$$\mathrm{R}^{-p}(A) = \mathrm{R}(A^{-p*}),$$

$$\mathrm{R}^{+p}(A) = \{\alpha + A(p,*) \mid \alpha \in \mathrm{R}^{-p}(A)\},$$

and $\mathrm{r}^{-p}(A) = |\mathrm{R}^{-p}(A)|$, $\mathrm{r}^{+p}(A) = |\mathrm{R}^{+p}(A)|$.

We say that $A \in \mathcal{B}_{p,q}$ is **submatrix** of $B \in \mathcal{B}_{m,n}$, $p \le m$, $q \le n$, and write $A \subset B$, if there exists $B' \equiv B$ such that

$$B' = \begin{bmatrix} A & A_1 \\ A_2 & A_3 \end{bmatrix}. \qquad (2)$$

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i.e. $A(i,j) = B'(i,j)$ for all $1 \le i \le p$, $1 \le j \le q$.

The weight $\mathrm{w}(\alpha)$ of $\alpha \in \mathcal{B}_{1,n}$ or $\alpha \in \mathcal{B}_{n,1}$ is the number of elements 1 in α. Note that if A and B are both row vectors (or column vectors), then $A \subset B$ implies $\mathrm{w}(A) \le \mathrm{w}(B)$. The transpose matrix of A is denoted by A^T.

Definition 1. If $A \in \mathcal{B}_{n,m}$ and if $(i_1, i_2, \ldots, i_n)$, $(j_1, j_2, \ldots, j_m)$ are the permutations of $(1,2,\ldots,n)$, $(1,2,\ldots,m)$ such that

$$\mathrm{w}(A(i_1,*)) \ge \mathrm{w}(A(i_2,*)) \ge \cdots \ge \mathrm{w}(A(i_n,*)),$$

$$\mathrm{w}(A(*,j_1)) \ge \mathrm{w}(A(*,j_2)) \ge \cdots \ge \mathrm{w}(A(*,j_m)),$$

respectively, then let

$$\mathrm{rws}(A,k) = \mathrm{w}(A(i_k,*)), \quad k = 1,2,\ldots,n$$

and

$$\mathrm{cws}(A,k) = \mathrm{w}(A(*,j_k)), \quad k = 1,2,\ldots,m$$

denote the *row weights spectrum* and *column weights spectrum* of A, respectively.

The following simple facts can be found in [6], for example.

Proposition 1. *Let $A \in \mathcal{B}_{n,m}$ and $p \ne q$. Then*

1) *If* $A \equiv B$ *then* $\mathrm{r}(A) = \mathrm{r}(B)$.
2) $\mathrm{r}(A) = \mathrm{r}(A^T)$.
3) $\mathrm{r}(A) \le \min\{2^n, 2^m\}$.
4) *If* $\mathrm{w}(A(p,*)) = k$ *then* $\mathrm{r}^{+p}(A) \le 2^{n-k}$.
5) $\mathrm{R}(A) = \mathrm{R}^{+p}(A) \cup \mathrm{R}^{-p}(A)$.
6) $\mathrm{r}(A) \le \mathrm{r}^{+p}(A) + \mathrm{r}^{-p}(A)$.
7) $\mathrm{R}^{-p}(A) = \mathrm{R}^{-p+q}(A) \cup \mathrm{R}^{-p-q}(A)$.
8) $\mathrm{R}(A) = \mathrm{R}^{+p}(A) \cup \mathrm{R}^{-p+q}(A) \cup \mathrm{R}^{-p-q}(A)$.
9) $\mathrm{r}(A) \le \mathrm{r}^{+p}(A) + \mathrm{r}^{-p+q}(A) + \mathrm{r}^{-p-q}(A)$.

2. Dense and Proper Matrices

We say that:

1) $A \in \mathcal{B}_n$ is **dense** if $\rho(A) > \frac{3}{8}$,
2) A is **proper** if there exists a pair (i,j), $1 \le i,j \le n$, such that $\rho(A^{-i*,-*j}) \ge \rho(A)$.

After establishing some useful facts, we prove Theorem 1, stating that all dense matrices are proper.

Lemma 1. *Let $k,l,m \ge 1$, and let*

$$B = \begin{bmatrix} 1 & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & A' & A \end{bmatrix} \in \mathcal{B}_{m+1,k+l+1},$$

where $A \in \mathcal{B}_{m,k}$ and $A' \in \mathcal{B}_{m,l}$. Then $\mathrm{r}(B) = \mathrm{r}\begin{bmatrix} A' & A \end{bmatrix} + \mathrm{r}(A)$.

Proof. All the elements of $\mathrm{R}^{+1}(B)$ start with 1_{l+1}, and so $\mathrm{r}^{+1}(B) = \mathrm{r}(A)$. Since all elements of $\mathrm{R}^{-1}(B)$ start with 0 and all elements of $\mathrm{R}^{+1}(B)$ start with 1, we have $\mathrm{R}^{-1}(B) \cap \mathrm{R}^{+1}(B) = \emptyset$, and according to Proposition 1.7)

$$\mathrm{r}(B) = \mathrm{r}^{-1}(B) + \mathrm{r}^{+1}(B) = \mathrm{r}\begin{bmatrix} A' & A \end{bmatrix} + \mathrm{r}(A).$$

□

Following Konieczny [2], denote by λ_n, ω_n and θ_n, $n \ge 1$, the matrices defined by

$$\lambda_n(i,j) = \begin{cases} 1, & j = i, \quad 1 \le i \le n-1 \\ & j = i+1, \quad 1 \le i \le n-1 \\ 0, & \text{otherwise} \end{cases}$$

$$\omega_n(i,j) = \begin{cases} 1, & j = i, \quad 1 \le i \le n \\ & j = i+1, \quad 1 \le i \le n-1 \\ 0, & \text{otherwise} \end{cases}$$

$$\theta_n(i,j) = \begin{cases} 1, & j = i, \quad 1 \le i \le n \\ & j = i+1, \quad 1 \le i \le n-1 \\ & j = 1, \quad i = n \\ 0, & \text{otherwise} \end{cases}$$

These sequences appear in [10] as the sequences A005251, A005314 and A259967, respectively. The following table lists $\mathrm{r}(\lambda_n)$, $\mathrm{r}(\omega_n)$ and $\mathrm{r}(\theta_n)$ for $n \le 12$.

n	1	2	3	4	5	6
$\mathrm{r}(\lambda_n)$	1	2	4	7	12	21
$\mathrm{r}(\omega_n)$	2	3	5	9	16	28
$\mathrm{r}(\theta_n)$	2	2	5	10	17	29
n	7	8	9	10	11	12
$\mathrm{r}(\lambda_n)$	37	65	114	200	351	616
$\mathrm{r}(\omega_n)$	49	86	151	265	465	816
$\mathrm{r}(\theta_n)$	51	90	158	277	486	853

Let E denote the *shift* operator, $Ef(n) = f(n+1)$. For example, $(E^2 - E + 1)\,\mathrm{r}(\omega_n) = \mathrm{r}(\omega_{n+2}) - \mathrm{r}(\omega_{n+1}) + \mathrm{r}(\omega_n)$.

Lemma 2. *Let*

$$g(E) = E^3 - 2E^2 + E - 1. \tag{3}$$

Then

$$g(E)\,\mathrm{r}(\lambda_n) = g(E)\,\mathrm{r}(\omega_n) = g(E)\,\mathrm{r}(\theta_n) = 0, \quad n > 0 \tag{4}$$

$$\mathrm{r}(\theta_n) < \mathrm{r}(\lambda_n) + \mathrm{r}(\lambda_{n-1}) < 4\,\mathrm{r}(\omega_{n-2}), \quad n > 2 \tag{5}$$

Proof. By Lemma 1 we obtain $\mathrm{r}(\lambda_n) = \mathrm{r}(\lambda_{n-1}) + \mathrm{r}(\omega_{n-2})$, $n \ge 3$, and

$$\mathrm{r}(\omega_n) = \mathrm{r}(\omega_{n-1}) + \mathrm{r}(\lambda_{n-1}), n \ge 3 \tag{6}$$

(it is obviously true if $n = 2$, also), i.e.

$$\mathrm{r}(\omega_n) = (E^2 - E)\,\mathrm{r}(\lambda_n), n \ge 1, \tag{7}$$

$$\mathrm{r}(\lambda_n) = (E - 1)\,\mathrm{r}(\omega_n), n \ge 1. \tag{8}$$

Substituting $\mathrm{r}(\omega_n)$ (7) into (8), we obtain

$$\mathrm{r}(\lambda_n) = (E-1)(E^2 - E)\,\mathrm{r}(\lambda_n), n \ge 1,$$

i.e.,

$$g(E)\,\mathrm{r}(\lambda_n) = 0, \quad n \ge 1. \tag{9}$$

From (7) it follows

$$g(E)\,\mathrm{r}(\omega_n) = (E^2 - E) g(E)\,\mathrm{r}(\lambda_n) = 0, \quad n \ge 1, \tag{10}$$

By using the Proposition 1.8) we get the following representation:

$$\begin{aligned}\mathrm{R}(\theta_n) = \Big(\mathrm{R}^{+n*}(\theta_n) \cup \mathrm{R}^{-n*,+(n-1)*,+1*}(\theta_n)\Big)\\ \cup\, \mathrm{R}^{-n*,-(n-1)*,+1*}(\theta_n) \cup \mathrm{R}^{-n*,-(n-1)*,-1*}(\theta_n)\\ \cup\, \mathrm{R}^{-n*,+(n-1)*,-1*}(\theta_n),\ n \geq 4.\end{aligned}$$

The pairs at positions 1, n in the elements of these four sets have values $(1,1)$, $(1,0)$, $(0,0)$, $(0,1)$ respectively, so the sets are disjoint. Furthermore, $\mathrm{R}^{-n*,+(n-1)*,+1*}(\theta_n) \subseteq \mathrm{R}^{+n*}(\theta_n)$, $n \geq 4$. Therefore,

$$\begin{aligned}\mathrm{r}(\theta_n) &= |\,\mathrm{R}^{+n*}(\theta_n)| + |\,\mathrm{R}^{-n*,-(n-1)*,+1*}(\theta_n)|\\ &\quad + |\,\mathrm{R}^{-n*,-(n-1)*,-1*}(\theta_n)| + |\,\mathrm{R}^{-n*,+(n-1)*,-1*}(\theta_n)|\\ &= \mathrm{r}(\lambda_{n-1}) + \mathrm{r}(\omega_{n-3}) + \mathrm{r}(\lambda_{n-2}) + \mathrm{r}(\omega_{n-3})\end{aligned}$$

Substituting $\mathrm{r}(\omega_n) = \mathrm{r}(\lambda_{n+2}) - \mathrm{r}(\lambda_{n+1})$ (7) into $\mathrm{r}(\theta_{n+3}) = \mathrm{r}(\lambda_{n+2}) + \mathrm{r}(\omega_n) + \mathrm{r}(\lambda_{n+1}) + \mathrm{r}(\omega_n)$, we obtain

$$\begin{aligned}\mathrm{r}(\theta_{n+3}) &= \mathrm{r}(\lambda_{n+3}) + \mathrm{r}(\lambda_{n+2}) - \mathrm{r}(\lambda_n)\\ &= (E^3 + E^2 - 1)\,\mathrm{r}(\lambda_n),\ n \geq 1.\end{aligned}$$

Multiplying with $g(E)$, we obtain $g(E)\,\mathrm{r}(\theta_n) = 0, \quad n \geq 4$. It is immediately seen that this is true also for $n = 1, 2, 3$, thus proving

$$g(E)\,\mathrm{r}(\theta_n) = 0, \quad n \geq 1,$$

which, together with (9) and (10), implies (4). From the identity

$$(E^2 + 2E + 2)g(E) = (E^5 - E^3 - 3E^2 - 2)$$

it follows

$$\begin{aligned}(E^5 - E^3 - 3E^2 - 2)\,\mathrm{r}(\omega_n) &= (E^2 + 2E + 2)g(E)\,\mathrm{r}(\omega_n)\\ &= 0,\ n \geq 1.\end{aligned}$$

But

$$\begin{aligned}(E^5 - E^3)\,\mathrm{r}(\omega_n) &= (E^4 + E^3)(E - 1)\,\mathrm{r}(\omega_n)\\ &= (E^4 + E^3)\,\mathrm{r}(\lambda_n),\end{aligned}$$

implying

$$\mathrm{r}(\lambda_{n+4}) + \mathrm{r}(\lambda_{n+3}) = 3\,\mathrm{r}(\omega_{n+2}) + 2\,\mathrm{r}(\omega_n),\ n \geq 1,$$

From (6) follows that $\mathrm{r}(\omega_n - 1) > \mathrm{r}(\omega_{n-2})$ and since $\mathrm{r}(\omega_n) = \mathrm{r}(\omega_{n-1}) + \mathrm{r}(\omega_{n-2}) + \mathrm{r}(\omega_{n-4})$ we have that $\mathrm{r}(\omega_n) > 2\,\mathrm{r}(\omega_{n-2})$. Combining this with the previous equality, we get $\mathrm{r}(\lambda_{n+4}) + \mathrm{r}(\lambda_{n+3}) < 4\,\mathrm{r}(\omega_{n+2})$, which is the second inequality of (5).

The first inequality of (5) follows from

$$\mathrm{r}(\theta_n) = \mathrm{r}(\lambda_n) + \mathrm{r}(\lambda_{n-1}) - \mathrm{r}(\lambda_{n-3}) < \mathrm{r}(\lambda_n) + \mathrm{r}(\lambda_{n-1}).$$

□

Definition 2. Let $B \in \mathcal{B}_n$, $A \subseteq B$ and $A \in \mathcal{B}_{k,l}$. We call A a **block** in B if for some $B' \equiv B$, we have

$$B' = \begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix},$$

and either $A = I_1$ or

1) for every row i, $1 \leq i \leq k$ there exists j, $1 \leq j \leq k$, $i \neq j$, and p, $1 \leq p \leq l$, such that $A(i,p) = A(j,p) = 1$;
2) for every column i, $1 \leq i \leq l$, there exists j, $1 \leq j \leq l$, $i \neq j$, and q, $1 \leq q \leq k$, such that $A(q,i) = A(q,j) = 1$.

Definition 3. Let $A \in \mathcal{B}_n$ be a matrix without zero rows and without zero columns. We say that matrix A is in **block diagonal form** if

$$A = \begin{bmatrix} A_1 & 0 & 0 & \dots & 0 \\ 0 & A_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & \mathcal{A}_{m-1} & 0 \\ 0 & \dots & 0 & 0 & A_m \end{bmatrix} \qquad (11)$$

where $A_i \in \mathcal{B}_{k_i,l_i}$ are blocks and $\sum k_i = \sum l_i = n$, $k_i, l_i \geq 1, 1 \leq i \leq m$. If $m = 1$ then we call A a **block matrix**. If there exists k, such that $A_k = I_1$ we say that A contains 1**-block**.

Let $B \in \mathcal{B}_n$ be a matrix without zero rows and without zero columns. Let $G = (V_1, V_2, E)$ be a bipartite graph with vertex sets $V_1 = \{a_1, a_2, \dots, a_n\}$ and $V_2 = \{b_1, b_2, \dots, b_n\}$, and with the set of edges $E = \{(a_i, b_j) | A(i,j) = 1\}$. Connected component $G_k = (V_{k_1}, V_{k_2}, E_k)$ of graph G corresponds to the block A_k of A, $1 \leq k \leq m$. Since $\mathrm{w}(A(i,*)) \geq 1$, $1 \leq i \leq n$, and $\mathrm{w}(A(*,j)) \geq 1$, $1 \leq j \leq n$, graph G does not contain isolated vertices. The blocks A_k, $1 \leq k \leq m$, obtained when we exclude from matrix B all rows i such that $a_i \notin V_{k_1}(G)$, and all columns j, such that $b_j \notin V_{k_2}(G)$, correspond to the blocks of matrix $A \equiv B$, that is in block diagonal form. This implies the following lemma.

Lemma 3. *Let $B \in \mathcal{B}_n$ be an arbitrary matrix without zero rows and without zero columns. Then there exists $A \equiv B$, such that A is in block diagonal form.*

Lemma 4. *If diagonal blocks of B are A_i, then $\mathrm{r}(B) = \prod \mathrm{r}(A_i)$.*

Proof. The row space of B is a direct sum of row spaces of row-sets of B corresponding to diagonal blocks. □

Lemma 5. *If $A \in \mathcal{B}_k$ is one of diagonal blocks of $B \in \mathcal{B}_n$, and if A is proper, then B is proper.*

Proof. We can suppose WLOG (without loss of generality) that A is the first diagonal block of B. If $A = I_1$, then $2 \cdot \mathrm{r}(B^{-1*,-*1}) = \mathrm{r}(B)$, and B is proper. Suppose now $A \neq I_1$. Because A is proper, suppose WLOG that the matrix $C = A^{-1*,-*1}$ is such that $\mathrm{r}(A) \leq 2\,\mathrm{r}(C)$. From Lemma 4 it follows $\mathrm{r}(B) = a\ \mathrm{r}(A)$, where a is the product of row space cardinalities of other blocks of B. If $D = B^{-1*,-*1}$, then $\mathrm{r}(D) = a\ \mathrm{r}(C)$ and $\mathrm{r}(B) = a\ \mathrm{r}(A) \leq 2a\ \mathrm{r}(C) = 2\,\mathrm{r}(D)$, hence B is proper. □

Lemma 6. *The matrix θ_n is proper, $n \geq 2$.*

Proof. Let $B = \theta_n^{-2*,-*1} \in \mathcal{B}_{n-1}$. If $n = 2$ then $\rho(B) = 1 > \rho(\theta_2) = 1/2$. Otherwise if $n > 2$, then $\mathrm{r}(B) = 2\,\mathrm{r}(\omega_{n-2})$, because B consists of two blocks, I_1 and ω_{n-2}. From Lemma 2 it follows $\mathrm{r}(\theta_n) < 4\,\mathrm{r}(\omega_{n-2}) = 2\,\mathrm{r}(B)$, i.e. $\rho(B) > \rho(\theta_n)$. □

Lemma 7. *Let $A \in \mathcal{B}_n$, $n \geq 2$, be matrix such that the weights of all rows and columns of A are equal to 2, i.e.* $\mathrm{rws}(A, 1) = \mathrm{rws}(A, n) = \mathrm{cws}(A, 1) = \mathrm{cws}(A, n) = 2$. *Then A is proper.*

Proof. By Lemma 3, there exists a matrix $A' \equiv A$, the block diagonal form of A. Since all rows an columns in A' have weight 2, the blocks of A' are equivalent to some θ_{m_i}, $m_i > 1$. By Lemma 6 these blocks are proper, and from Lemma 5 it follows that A is proper. □

Lemma 8. *Let $A = (\lambda_n^{-n*})^T \in \mathcal{B}_{n,n-1}$. Let $\alpha \in \mathcal{B}_{n,1}$, $\mathrm{w}(\alpha) = 3$, $\alpha(1) = 1$, $\alpha(n) = 1$ and $\alpha(j) = 1$ for some $1 < j < n$. Then the matrix $B = \begin{bmatrix}\alpha & A\end{bmatrix}$ is proper.*

Proof. Let $C = A^{-(n-1)*} = B^{-(n-1)*,-*1}$; it is enough to prove that $\rho(C) \geq \rho(B)$. Let

$$D = \left[\begin{array}{c|c} \multicolumn{2}{c}{\theta_n} \\ \hline 1 & 0_{1,n-1} \end{array}\right] \in \mathcal{B}_{n+1,n}.$$

From $D^{-j*,-(n+1)*} = B^{-j*}$ it follows $\mathrm{R}^{-j*,-(n+1)*}(D) = \mathrm{R}^{-j*}(B)$. We also have that $\mathrm{R}^{+j*,+(n+1)*}(D) = \mathrm{R}^{+j*}(B)$. Hence $\mathrm{R}^{-j*}(B) \cup \mathrm{R}^{+j*}(B) = \mathrm{R}(B) \subseteq \mathrm{R}(D)$ and $\mathrm{r}(B) \leq \mathrm{r}(D)$. Applying Lemma 1 to transpose of D, we obtain $\mathrm{r}(D) = \mathrm{r}(\lambda_n) + \mathrm{r}(\lambda_{n-1})$, and so $\mathrm{r}(B) \leq \mathrm{r}(\lambda_n) + \mathrm{r}(\lambda_{n-1})$.

The matrix C has two diagonal blocks, and so $\mathrm{r}(C) = 2\,\mathrm{r}(\omega_{n-2})$. From Lemma 2 it follows

$$2\,\mathrm{r}(C) = 4\,\mathrm{r}(\omega_{n-2}) > \mathrm{r}(\lambda_n) + \mathrm{r}(\lambda_{n-1}) > \mathrm{r}(B),$$

thus proving $\rho(C) > \rho(B)$. □

Lemma 9. *Let $A \in \mathcal{B}_n$, $p + q = n$, and*

$$A(i,j) = \begin{cases} 1, & j = i,\ i = 1, 2, \ldots, n \\ & j = i+1,\ i = 1, 2, \ldots, n-1 \\ & i = p,\ j = 1 \\ & i = n,\ j = p+1 \\ 0, & \textit{otherwise} \end{cases}$$

i. e.

$$A = \left[\begin{array}{c|cc} \theta_p & \begin{matrix} 0 \\ I_1 \end{matrix} & \begin{matrix} \mathbf{0} \\ \mathbf{0} \end{matrix} \\ \hline \mathbf{0} & \multicolumn{2}{c}{\theta_q} \end{array}\right]$$

Then A is proper.

Proof. Let $C = A^{-(n-1)*,-*(p+1)} \in \mathcal{B}_{n-1}$; we prove that $\rho(C) \geq \rho(A)$. Let

$$D = \left[\begin{array}{c|cc} \theta_p & \multicolumn{2}{c}{\mathbf{0}} \\ \hline \mathbf{0} & \multicolumn{2}{c}{\theta_q} \\ \hline \mathbf{0} & I_1 & \mathbf{0} \end{array}\right] \in \mathcal{B}_{n+1,n}$$

Then $D^{-p*,-(n+1)*} = A^{-p*}$ imply $\mathrm{R}^{-p*,-(n+1)*}(D) = \mathrm{R}^{-p*}(A)$. We also have that $\mathrm{R}^{+p*,+(n+1)*}(D) = \mathrm{R}^{+p*}(A)$, and consequently $\mathrm{R}^{-p*}(A) \cup \mathrm{R}^{+p*}(A) = \mathrm{R}(A) \subseteq \mathrm{R}(D)$ and $\mathrm{r}(A) \leq \mathrm{r}(D)$. By Lemma 1 we have

$$\mathrm{r}(D) = \mathrm{r}(\theta_p)(\mathrm{r}(\lambda_q) + \mathrm{r}(\lambda_{q-1})),$$

and so

$$\mathrm{r}(A) \leq \mathrm{r}(\theta_p)(\mathrm{r}(\lambda_q) + \mathrm{r}(\lambda_{q-1})).$$

Furthermore $\mathrm{r}(C) = \mathrm{r}(\theta_p) \cdot 2\,\mathrm{r}(\omega_{q-2})$, hence by Lemma 2 we have

$$2\,\mathrm{r}(C) = 4\,\mathrm{r}(\theta_p)\,\mathrm{r}(\omega_{q-2}) > \mathrm{r}(A)$$

and $\rho(C) > \rho(A)$. □

Lemma 10. *Let $a \in \{0, 1\}$, $\alpha \in \mathcal{B}_{1,n-1}$, $\beta \in \mathcal{B}_{n-1,1}$, $A' \in \mathcal{B}_{n-1}$ and let*

$$A = \begin{bmatrix} a & \alpha \\ \beta & A' \end{bmatrix}$$

If $\mathrm{w}(\beta) \geq 3$ and $\mathrm{w}(\alpha) + a \geq 3$, or $\mathrm{w}(\beta) \geq 2$ and $\mathrm{w}(\alpha) + a \geq 4$ then

a) $\mathrm{r}(A) \leq \mathrm{r}(A') + 2^{n-3} + 2^{n-4}$.

b) *If A is dense, then A' is dense and A is proper.*

Proof. a) Let $k = \mathrm{w}(\beta)$, $l = \mathrm{w}(\alpha) + a$. Since $A' \in \mathcal{B}_{n-1}$, from Proposition 1 it follows

$$r\begin{bmatrix}\beta & A'\end{bmatrix} \leq \mathrm{r}(A') + 2^{n-1-k}$$

and

$$\mathrm{r}(A) \leq \mathrm{r}(A') + 2^{n-1-k} + 2^{n-l}.$$

Since $k \geq 3$ and $l \geq 3$, or $k \geq 2$ and $l \geq 4$, we have in both cases $\mathrm{r}(A) \leq \mathrm{r}(A') + 2^{n-3} + 2^{n-4}$.

b) If $\rho(A) > \frac{3}{8}$, i.e. $\mathrm{r}(A) > 2^{n-2} + 2^{n-3}$, then $\mathrm{r}(A') \geq \mathrm{r}(A) - 2^{n-3} - 2^{n-4} > 2^{n-3} + 2^{n-4}$, hence $\mathrm{r}(A) \leq \mathrm{r}(A') + 2^{n-3} + 2^{n-4} < 2\,\mathrm{r}(A')$, i.e. $\rho(A') > \rho(A) > \frac{3}{8}$. □

Lemma 11. *Suppose $A \in \mathcal{B}_{n,m}$, $\alpha = A(n, *)$ and $\mathrm{w}(\alpha) = k$. Furthermore, suppose there exists $\beta \in \mathrm{R}^{-n*}(A)$, such that $\alpha + \beta = \beta$ (i.e. if $\alpha(i) = 1$ then also $\beta(i) = 1$, for every $1 \leq i \leq m$), and $\mathrm{w}(\beta) = l \geq k$. Then $\mathrm{r}(A) \leq \mathrm{r}^{-n*}(A) + 2^{n-k} - 2^{n-l}$. The analogous inequality holds for the column space cardinalities.*

Proof. If $l = k$, then $\alpha \in \mathrm{R}^{-n*}(A)$, and $\mathrm{r}(A) = \mathrm{r}^{-n*}(A)$, so suppose $l > k$. Let $\gamma \in \mathrm{R}^{+n*}(A) \setminus \mathrm{R}^{-n*}(A)$. Then $\alpha + \gamma = \gamma$, i.e. if $\alpha(i) = 1$ then also $\gamma(i) = 1$, for every $1 \leq i \leq m$. Furthermore, γ cannot contain all the ones contained in β and not contained in α — otherwise it would be $\gamma \in \mathrm{R}^{-n*}(A)$. The number of vectors γ that satisfy these two conditions is $2^{n-l}(2^{l-k} - 1) = 2^{n-k} - 2^{n-l}$. From Proposition 1 follows that $\mathrm{r}(A) \leq \mathrm{r}^{-n*}(A) + 2^{n-k} - 2^{n-l}$. □

Note that this is somewhat better upper bound for $\mathrm{r}(A)$, than the simple bound $\mathrm{r}(A) \leq \mathrm{r}^{-n*}(A) + 2^{n-k}$, that follows from Proposition 1.

Lemma 12. *Let $A \in \mathcal{B}_n$, is dense, $n > 3$, and suppose*

$$A = \left[\begin{array}{c|ccc|c} 1 & 1 & 0_{1,n-4} & 0 & 1 \\ \hline 1 & & & & 0 \\ 0_{n-4,1} & & A' & & 0_{n-4,1} \\ 0 & & & & 1 \\ \hline 1 & 0 & 0_{1,n-4} & 1 & 1 \end{array}\right]$$

where $\mathrm{w}(A(i,*)) = 2$, $1 < i < n$, *and* $\mathrm{w}(A(*,j)) = 2$, $1 < j < n$. *Then A is proper.*

Proof. Let $B = A^{-n*,-*n}$, $D = A^{-*n}$ and $\alpha = D(n,*)$. We have $\mathrm{w}(\alpha) = 2$, $\alpha(1) = \alpha(n-1) = 1$. Since $\mathrm{w}(A(*,n-1)) = 2$, we have $\mathrm{w}(B(*,n-1)) = 1$, implying $B(j,n-1) = 1$ for some j, $2 \le j \le n-1$. Let $\gamma = B(1,*) + B(j,*) \in \mathrm{R}(B)$. Depending on the value of j, we have two cases:

1) $j \in \{2, n-1\}$. Then $\mathrm{w}(\gamma) = 3$.
2) $2 < j < n-1$. Then $\mathrm{w}(B(j,*)) = 2$, and $B(j,i) = 1$ for some i, $2 \le i \le n-1$, and $\mathrm{w}(\gamma) \le \mathrm{w}(D(1,*)) + \mathrm{w}(D(j,*)) = 4$.

From Lemma 11 it follows

$$\mathrm{r}(D) \le \mathrm{r}(B) + 2^{(n-1)-2} - 2^{(n-1)-4} = \mathrm{r}(B) + 2^{n-4} + 2^{n-5}.$$

Since $\mathrm{w}(A(n-1,*)) = 2$, we have $\mathrm{w}(D(n-1,*)) = 1$, implying $D(n-1,j) = 1$ for some j, $2 \le j \le n-1$. Let $\gamma = D(*,1) + D(*,j) \in \mathrm{C}(D)$. Depending on the value of j, we have two cases:

1) $j \in \{2, n-1\}$. Then $\mathrm{w}(\gamma) = 4$.
2) $2 < j < n-1$. Then $\mathrm{w}(D(*,j)) = 2$, and $D(i,j) = 1$ for some i, $2 \le i \le n-1$, and $\mathrm{w}(\gamma) \le \mathrm{w}(D(*,1)) + \mathrm{w}(D(*,j) = 5$.

From Lemma 11 it follows

$$|\,\mathrm{C}(A)| \le |\,\mathrm{C}(D)| + 2^{n-3} - 2^{n-5} = |\,\mathrm{C}(D)| + 2^{n-4} + 2^{n-5}.$$

Combining this with the upper bound on $\mathrm{r}(D)$, we obtain

$$\mathrm{r}(A) \le \mathrm{r}(B) + 2^{n-4} + 2^{n-5} + 2^{n-4} + 2^{n-5} = \mathrm{r}(B) + 2^{n-3} + 2^{n-4}.$$

Since A is dense, i.e. $\mathrm{r}(A) > 2^{n-2} + 2^{n-3}$, we have

$$2^{n-2} + 2^{n-3} < \mathrm{r}(B) + 2^{n-3} + 2^{n-4},$$

hence $\mathrm{r}(B) > 2^{n-3} + 2^{n-4}$, so

$$\mathrm{r}(A) \le \mathrm{r}(B) + 2^{n-3} + 2^{n-4} < 2 \cdot \mathrm{r}(B),$$

implying that A is proper. ☐

Lemma 13. *If*

$$A = \begin{bmatrix} B & 0_{n-1,1} \\ \alpha & a \end{bmatrix} \in \mathcal{B}_n.$$

then $\rho(B) \ge \rho(A)$,

Proof. From $\mathrm{r}\begin{bmatrix} B & 0_{n-1,1} \end{bmatrix} = \mathrm{r}(B)$, it follows $\mathrm{r}(A) \le 2\,\mathrm{r}(B)$ ☐

Definition 4. We say that matrix A is **redundant** if there exists a row (column) i such that $\mathrm{r}(A) = \mathrm{r}(A^{-i*})$ ($\mathrm{r}(A) = \mathrm{r}(A^{-*i})$).

Lemma 14. *Every redundant matrix is proper.*

Proof. Let A be redundant matrix such that $\mathrm{r}(A) = \mathrm{r}(A^{-i*})$, and let $1 \le j \le n$ be an arbitrary column index. Then $\mathrm{r}(A) \le 2\,\mathrm{r}(A^{-i*,-*j})$, i.e. $\rho(A) \le \rho(A^{-i*,-*j})$. ☐

Theorem 1. *Let $A \in \mathcal{B}_n$. If A is dense and $n \ge 2$, then A is proper.*

Proof. Let

$$m = \min\{\mathrm{rws}(A,n), \mathrm{cws}(A,n)\}$$

and

$$M = \max\{\mathrm{rws}(A,1), \mathrm{cws}(A,1)\}$$

Case $m \le 1$:
then A contains a row or column with weight in $\{0,1\}$, and the claim follows from Lemma 13.

Case $m = M = 2$:
then A is proper by Lemma 7.

Case $m \ge 2$ and $M \ge 4$:
then there exists k, such that $\mathrm{w}(A(k,*)) \ge 4$ (or $\mathrm{w}(A(*,k)) \ge 4$), and because the sum of rows of A equals to the sum of columns of A, there exists l, such that $\mathrm{w}(A(*,l)) \ge 3$ ($\mathrm{w}(A(l,*)) \ge 3$). From Lemma 10 it follows that A is proper.

Case $m \in \{2,3\}$ and $M = 3$:
Let $t > 0$ denote the number of rows with the weight equal to 3. Assume WLOG that $\mathrm{w}(A(1,*)) = 3$. There exists k, such that $\mathrm{w}(A(*,k)) = 3$ (otherwise would be $t = 0$ and $M < 3$). If $A(1,k) = 0$ (or more generally, if the element of A at the intersection of any row with the weight 3 with any column with the weight 3 is 0), then from Lemma 10 it follows that $\rho(B) \ge \rho(A)$, where B is the matrix obtained from A by removing the corresponding row and column with the weight equal to 3.
Therefore, we suppose that any row with weight 3 and any column with weight 3 intersect at the element of A with the value equal to 1.

Case $t = 1$:
Only the first row and the kth column of A have weight 3, and from previous assumption we have that $A(1,k) = 1$. If A contains a block equivalent to θ_l as a submatrix, then by Lemma 6 there exists $C \in \mathcal{B}_{l-1}$, such that $\rho(C) \ge \rho(\theta_l)$, so the claim is true. Otherwise, A is a block matrix, so it's equivalent to either matrix from Lemma 8 or Lemma 9, hence it is proper.

Case $t \ge 2$:
We can assume that A doesn't contain two identical rows or columns, since otherwise it's redundant, and consequently it's proper. If $t = 2$, since $M = 3$, there are exactly two rows and two columns of weight 3. Four elements at their intersections are all 1s (otherwise the claim follows from Lemma 10), therefore from

Lemma 12 it follows that the claim is true. If $t > 2$, A contains two identical rows (columns), or it contains row i and column j, with weight 3, such that $A(i,j) = 0$, and in both cases A is proper. □

Corrolary 1. *Let $A_n \in \mathcal{B}_n$, $n \geq 2$, and A_n is dense. Then there exists a sequence A_1, A_2, ..., A_{n-1}, such that $A_i \in \mathcal{B}_i$, $A_{i-1} \subset A_i$ and $\rho(A_{i-1}) \geq \rho(A_i)$ for each $i = 2, , \ldots, n$.*

We verified with computer that for $2 \leq n \leq 7$ all matrices in $\mathcal{B}_n$ are proper, so we state the following conjecture.

Conjecture 1. *Let $n \geq 2$ and $A \in \mathcal{B}_n$. Then A is proper.*

Lemma 15. *Let $A \in \mathcal{B}_n$, A has two rows (column) with weights not less than 4, i.e.* $\mathrm{rws}(A,2) \geq 4$ *or* $\mathrm{cws}(A,2) \geq 4$. *Then* $\rho(A) \leq \frac{3}{8}$.

Proof. We may let $\mathrm{w}(A(1,*)) \geq 4$ and $\mathrm{w}(A(2,*)) \geq 4$. Then $\mathrm{r}^{+1}(A) \leq 2^{n-4}$, $\mathrm{r}^{-1,+2}(A) \leq 2^{n-4}$ (from Proposition 1), and we have $\mathrm{r}^{-1,-2}(A) \leq 2^{n-2}$. We have, from Proposition 1:

$$\mathrm{r}(A) \leq 2^{n-4} + 2^{n-4} + 2^{n-2} = 2^{n-2} + 2^{n-3}$$

so $\rho(A) \leq \frac{3}{8}$. The proof for the case when two columns have weights not less than 4 is analogous. □

Definition 5. Suppose $A \in \mathcal{B}_n$. Define $\mathrm{type}(A)$ as follows:

$$\mathrm{type}(A) = \begin{cases} 0 & \text{if } \mathrm{rws}(A,1) \leq 4,\ \mathrm{rws}(A,2) \leq 3, \\ & \quad \mathrm{cws}(A,1) \leq 3 \\ & \text{or } \mathrm{cws}(A,1) \leq 4,\ \mathrm{cws}(A,2) \leq 3, \\ & \quad \mathrm{rws}(A,1) \leq 3 \\ 1 & \text{if } \mathrm{cws}(A,1) > 4,\ \mathrm{cws}(A,2) \leq 3, \\ & \quad \mathrm{rws}(A,1) \leq 3 \\ & \text{or } \mathrm{rws}(A,1) > 4,\ \mathrm{rws}(A,2) \leq 3, \\ & \quad \mathrm{cws}(A,1) \leq 3 \\ 2 & \text{if } \mathrm{rws}(A,1) > 3,\ \mathrm{cws}(A,1) > 3, \\ & \quad \mathrm{rws}(A,2) > 3,\ \mathrm{cws}(A,2) \leq 3, \\ 3 & \text{otherwise} \end{cases}$$

Lemma 16. *Let $A \in \mathcal{B}_n$. If A is dense, then* $\mathrm{type}(A) \in \{0,1,2\}$.

Proof. Obviously, $\mathrm{type}(A)$ is uniquely defined for all $A \in \mathcal{B}_n$. Since A is dense, from Lemma 15 it follows that $\mathrm{rws}(A,2) \leq 3$ and $\mathrm{cws}(A,2) \leq 3$, hence $\mathrm{type}(A) \in \{0,1,2\}$. □

Definition 6. Denote by $\mathcal{D}$ the class of dense matrices $\{A \in \mathcal{B}_n \mid n \geq 1, A \text{ is dense}\}$. Denote $\mathcal{D}_i = \{A \in \mathcal{D} \mid \mathrm{type}(A) = i\}$, $i = 0,1,2$.

Obviously,

$$\mathcal{D} = \mathcal{D}_0 \cup \mathcal{D}_1 \cup \mathcal{D}_2.$$

Definition 7. Let $A \in \mathcal{B}_n$, and let $1 \leq i,j,k,l \leq n$, $i \neq k$, $j \neq l$, $x,y \in \{0,1\}$. We say that A has a **cross** $\mathcal{K}_{x,y}(A,i,j,k,l)$ if $A(k,j) = x$, $A(i,l) = y$, $A(k,l) = 1$, $\mathrm{w}(A(k,*)) = x+1$ and $\mathrm{w}(A(*,l)) = y+1$.

If $x = 0$, then in $\mathcal{K}_{x,y}(A,i,j,k,l)$ we write $*$ instead of j, $\mathcal{K}_{0,y}(A,i,*,k,l)$, because $A(k,p) = 0$ for all $p \neq l$. Similarly, if $y = 0$, then we write $*$ instead of i: $\mathcal{K}_{x,0}(A,*,j,k,l)$. The element $A(k,l) = 1$ is 1-block if and only if A has a cross $\mathcal{K}_{0,0}(A,*,*,k,l)$.

Definition 8. Let $B \in \mathcal{B}_n \cap \mathcal{D}$. We say that A is a **base** matrix of B and denote $A = \mathrm{Base}(B)$ if A is obtained from B by Algorithm 1.

Algorithm 1. Given $B \in \mathcal{B}_n \cap \mathcal{D}$, compute $A = \mathrm{Base}(B)$.

Input : $B \in \mathcal{B}_n$;
Output : $A = \mathrm{Base}(B) \in \mathcal{B}_p$, $p \leq n$.
begin
  while B has a 1-block $\mathcal{K}_{0,0}(*,*,k,l)$
    and $B \in \mathcal{B}_m$, $m > 1$ **do**
    remove row k and column l from B.
  if $\mathrm{type}(B) = 1$ **then**
    we can assume that $\mathrm{w}(B(*,1)) > 4$.
    while $\mathrm{w}(B(*,1)) > 5$
      and B has a cross $\mathcal{K}_{1,0}(B,*,1,k,l)$ **do**
      remove row k and column l from B.
  else if $\mathrm{type}(B) = 2$ **then**
    we can assume that $\mathrm{w}(B(1,*)) > 3$ and $\mathrm{w}(B(*,1)) > 3$.
    while $\mathrm{w}(B(*,1)) > 4$
      and B has a cross $\mathcal{K}_{1,0}(B,*,1,k,l)$ **do**
      remove row k and column l from B.
    while $\mathrm{w}(B(1,*)) > 4$
      and B has a cross $\mathcal{K}_{0,1}(B,1,*,k,l)$ **do**
      remove row k and column l from B.
    while $\mathrm{w}(B(*,1)) > 4$ **and** $\mathrm{w}(B(1,*)) > 4$
      and B has crosses $\mathcal{K}_{1,1}(B,1,1,k_1,l_1)$
      and $\mathcal{K}_{1,1}(B,1,1,k_2,l_2)$,
      $k_1 \neq k_2$ and $l_1 \neq l_2$ **do**
        remove row k_2 and column l_2 from B.
  the result is the matrix $A = B \in \mathcal{B}_p$, $p \leq n$.
end

Note that $\mathrm{type}(B) = \mathrm{type}(\mathrm{Base}(B))$.

Let

$$A = \begin{bmatrix} a & \beta \\ \alpha & A' \end{bmatrix} \in \mathcal{B}_k,\ k \geq 1,$$

where $A' \in \mathcal{B}_{k-1}$, and $x,y \in \{0,1\}$. We denote by $\mathcal{X}_{x,y}(A)$ the matrix

$$\mathcal{X}_{x,y}(A) = \begin{bmatrix} a & \beta & y \\ \alpha & A' & 0 \\ x & 0 & 1 \end{bmatrix}$$

Obviously,

$$\mathcal{X}_{x_1,y_1}\mathcal{X}_{x_2,y_2}(A) \equiv \mathcal{X}_{x_2,y_2}\mathcal{X}_{x_1,y_1}(A). \tag{12}$$

We will use the following lemma [9, Lemma 3.2.]

Lemma 17. *If A has no zero rows and no zero columns, then*

$$\mathrm{r}\left[\begin{array}{c|cc} A & 0 & 1 \\ \hline 0 & C & D \\ 1 & E & F \end{array}\right] = (\mathrm{r}(A) - 2)\,\mathrm{r}(C) + \mathrm{r}\left[\begin{array}{c|cc} 1 & 0 & 1 \\ \hline 0 & C & D \\ 1 & E & F \end{array}\right]$$

Lemma 18. *Let $A \in \mathcal{B}_n$, $A' \in \mathcal{B}_{n-3}$, $\alpha \in \mathcal{B}_{n-3,1}$, $\beta \in \mathcal{B}_{1,n-3}$, $a \in \mathcal{B}_{1,1}$,*

$$A = \mathcal{X}_{1,1}^2 \begin{bmatrix} a & \beta \\ \alpha & A' \end{bmatrix} = \begin{bmatrix} a & \beta & 1 & 1 \\ \alpha & A' & \mathbf{0} & \mathbf{0} \\ 1 & \mathbf{0} & 1 & 0 \\ 1 & \mathbf{0} & 0 & 1 \end{bmatrix},$$

and $B = A^{-n*,-*n}$. Then $\rho(B) \geq \rho(A)$ and A is proper.

Proof. By Lemma 17 we have

$$\mathrm{r}(A) = (\mathrm{r}(I_2) - 2)\,\mathrm{r}(A') + \mathrm{r}(B) = 2\,\mathrm{r}(A') + \mathrm{r}(B)$$

Let $C = \begin{bmatrix} A' & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}$, then $\mathrm{r}(C) = 2\,\mathrm{r}(A')$. Since $C \subset B$ we have $2\,\mathrm{r}(A) = \mathrm{r}(C) \leq \mathrm{r}(B)$ and

$$\mathrm{r}(A) = 2\,\mathrm{r}(A') + \mathrm{r}(B) \leq 2\,\mathrm{r}(B)$$

which completes the proof. □

Lemma 19. *If $B \in \mathcal{D}_i$, then $A = \mathrm{Base}(B) \in \mathcal{D}_i$, $i = 0, 1, 2$.*

Proof. Since A is obtained from B by Algorithm 1 we have that $\mathrm{type}(B) = \mathrm{type}(A)$. Consider the following cases:

Case $B \in \mathcal{D}_0$:
Then $\rho(A) = \rho(B) > \frac{3}{8}$.
Case $B \in \mathcal{D}_1$:
Then, from Lemma 13 it follows $\rho(A) \geq \rho(B) > \frac{3}{8}$.
Case $B \in \mathcal{D}_2$:
Then, from Lemmas 13 and 18 it follows $\rho(A) \geq \rho(B) > \frac{3}{8}$.

□

Definition 9. For $i = 0, 1, 2$ define $\mathcal{E}_i$ by

$$\mathcal{E}_i = \bigcup_{B \in \mathcal{D}_i} \{\mathrm{can}(\mathrm{Base}(B))\}$$

Definition 10. Let $n \geq 1$ and $A \in \mathcal{B}_n \cap \mathcal{E}_0$. Define the class $\mathcal{C}_0(A)$ by

$$\mathcal{C}_0(A) = \left\{ \mathcal{X}_{0,0}^p(A) \mid p \geq 0 \right\}.$$

Definition 11. Let $A = \begin{bmatrix} \alpha & A' \end{bmatrix} \in \mathcal{B}_n \cap \mathcal{E}_1$ and suppose $\mathrm{w}(\alpha) > 4$. Define the class $\mathcal{C}_1(A)$ by

$$\mathcal{C}_1(A) = \left\{ \mathcal{X}_{1,0}^q \mathcal{X}_{0,0}^p(A) \mid p, q \geq 0 \right\}.$$

Definition 12. Let $A = \begin{bmatrix} a & \beta \\ \alpha & A' \end{bmatrix} \in \mathcal{B}_n \cap \mathcal{E}_2$, where $A' \in \mathcal{B}_{n-1,n-1}$, $\mathrm{w}(\alpha) + a > 3$ and $\mathrm{w}(\beta) + a > 3$. Define the class $\mathcal{C}_2(A)$ by

$$\mathcal{C}_2(A) = \left\{ \mathcal{X}_{1,1}^s \mathcal{X}_{0,1}^r \mathcal{X}_{1,0}^q \mathcal{X}_{0,0}^p(A) \mid p, q, r, s \geq 0 \right\}.$$

Proposition 2. *For each $i \in \{1, 2, 3\}$, and for each $D \in \mathcal{D}_i$, if $E = \mathrm{Base}(D)$, then $D \in \mathcal{C}_i(E)$.*

Proof. Follows from the fact that E is obtained from D by Algorithm 1. □

Let $A = \begin{bmatrix} a & \beta \\ \alpha & A' \end{bmatrix}$. We now show how $\mathrm{r}(B)$ can be computed if $B \in \mathcal{C}_i(A)$, $i = 0, 1, 2$. First we have

$$\mathrm{r}\left(\mathcal{X}_{0,0}^p(A)\right) = \mathrm{r} \begin{bmatrix} a & \beta & 0 \\ \alpha & A' & 0 \\ 0 & 0 & I_p \end{bmatrix} = 2^p\,\mathrm{r}(A), \quad p \geq 0$$

Using Lemma 17 and Lemma 1 we obtain

$$\begin{aligned} \mathrm{r}\left(\mathcal{X}_{1,0}^q(A)\right) &= (2^q - 2)\,\mathrm{r} \begin{bmatrix} \beta \\ A' \end{bmatrix} + \mathrm{r} \begin{bmatrix} a & \beta & 0 \\ \alpha & A' & 0 \\ 1 & 0 & 1 \end{bmatrix} \\ &= (2^q - 1)\,\mathrm{r} \begin{bmatrix} \beta \\ A' \end{bmatrix} + \mathrm{r} \begin{bmatrix} a & \beta \\ \alpha & A' \end{bmatrix}, \quad q \geq 0. \end{aligned} \tag{13}$$

Note that this equality trivially holds if $q = 0$. Analogously,

$$\begin{aligned} \mathrm{r}\left(\mathcal{X}_{0,1}^r(A)\right) &= (2^r - 2)\,\mathrm{r} \begin{bmatrix} \alpha & A' \end{bmatrix} + \mathrm{r} \begin{bmatrix} a & \beta & 1 \\ \alpha & A' & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= (2^r - 1)\,\mathrm{r} \begin{bmatrix} \alpha & A' \end{bmatrix} + \mathrm{r} \begin{bmatrix} a & \beta \\ \alpha & A' \end{bmatrix}, \quad r \geq 0. \end{aligned} \tag{14}$$

Theorem 2. *Let $A' \in \mathcal{B}_{k,k-1}$, $\alpha \in \mathcal{B}_{k,1}$, $A = [\alpha\ A']$, and let*

$$B = \mathcal{X}_{0,0}^p \mathcal{X}_{1,0}^q(A) = \begin{bmatrix} \alpha & A' & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & I_q & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & I_p \end{bmatrix} \in \mathcal{B}_n.$$

Then

$$\mathrm{r}(B) = 2^{n-k}\,\mathrm{r}(A') + 2^p\left(r[\alpha\ A'] - \mathrm{r}(A')\right). \tag{15}$$

Proof. Let

$$B' = \begin{bmatrix} \alpha & A' & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & I_q \end{bmatrix}$$

Using (13), we obtain $\mathrm{r}(B') = 2^q\,\mathrm{r}(A') + \mathrm{r}[\alpha\ A'] - \mathrm{r}(A')$, hence $\mathrm{r}(B) = 2^{p+q}\,\mathrm{r}(A') + 2^p(\mathrm{r}[\alpha\ A'] - \mathrm{r}(A'))$ proving (15), since $n = k + p + q$. □

The following theorem solves the general case.

Theorem 3. *Let $A' \in \mathcal{B}_{k-1,k-1}$, $A = \begin{bmatrix} a & \beta \\ \alpha & A' \end{bmatrix} \in \mathcal{B}_k$ Let $p, q, r, s \geq 0$, $B \in \mathcal{B}_n$, and let*

$$B = \mathcal{X}_{0,0}^p \mathcal{X}_{1,0}^q \mathcal{X}_{0,1}^r \mathcal{X}_{1,1}^s(A) = \begin{bmatrix} a & \beta & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \alpha & A' & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & I_s & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & I_r & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_q & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_p \end{bmatrix}$$

where $n = k + p + q + r + s$. Denote

$$A_2 = \begin{bmatrix} \alpha & A' \end{bmatrix}, \quad A_3 = \begin{bmatrix} \beta \\ A' \end{bmatrix}, \quad A_4 = \begin{bmatrix} a & \beta & 1 \\ \alpha & A' & \mathbf{0} \\ 1 & \mathbf{0} & 1 \end{bmatrix},$$

Then

$$\mathrm{r}(B) = 2^{n-k}\,\mathrm{r}(A') + 2^{p+q}c_3 + 2^{p+r}c_2 + 2^p c_1, \tag{16}$$

where

$$c_2 = \mathrm{r}(A_2) - \mathrm{r}(A'),\ c_3 = \mathrm{r}(A_3) - \mathrm{r}(A'),$$

and

$$c_1 = \begin{cases} \mathrm{r}(A) + \mathrm{r}(A') - \mathrm{r}(A_2) - \mathrm{r}(A_3), & s = 0 \\ \mathrm{r}(A_4) - \mathrm{r}(A_2) - \mathrm{r}(A_3), & s > 0. \end{cases} \tag{17}$$

Proof. Let

$$B' = \begin{bmatrix} a & \beta & \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \alpha & A' & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & I_s & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & I_r & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_q \end{bmatrix}$$

and suppose first $s = 0$. Using (13) we obtain

$$\mathrm{r}(B') = (2^q - 1) \begin{bmatrix} \beta & \mathbf{1} \\ A' & \mathbf{0} \\ \mathbf{0} & I_r \end{bmatrix} + \begin{bmatrix} a & \beta & \mathbf{1} \\ \alpha & A' & \mathbf{0} \\ 1 & \mathbf{0} & I_r \end{bmatrix}.$$

By (14) we have

$$\mathrm{r} \begin{bmatrix} \beta & \mathbf{1} \\ A' & \mathbf{0} \\ \mathbf{0} & I_r \end{bmatrix} = (2^r - 1)\,\mathrm{r}(A') + \mathrm{r}(A_3)$$

and

$$\mathrm{r} \begin{bmatrix} a & \beta & \mathbf{0} \\ \alpha & A' & \mathbf{0} \\ 1 & \mathbf{0} & I_r \end{bmatrix} = (2^r - 1)\,\mathrm{r}(A_2) + \mathrm{r}(A),$$

hence

$$\begin{aligned} \mathrm{r}(B') &= (2^q - 1)((2^r - 1)\,\mathrm{r}(A') + \mathrm{r}(A_3)) \\ &\quad + (2^r - 1)\,\mathrm{r}(A_2) + \mathrm{r}(A) \\ &= 2^{q+r}\,\mathrm{r}(A') + 2^r c_2 + 2^q c_3 + c_1. \end{aligned}$$

Multiplying this equality with 2^p and replacing $p + q + r$ with $n - k$, we obtain (16) for the case $s = 0$.

In order to obtain $\mathrm{r}(B')$ if $s > 0$, we use (16) for the case $s = 0$ and replace there

$$A',\ \begin{bmatrix} \alpha & A' \end{bmatrix},\ \begin{bmatrix} \beta \\ A' \end{bmatrix},\ \begin{bmatrix} a & \beta \\ \alpha & A' \end{bmatrix}$$

with

$$\begin{bmatrix} A' & 0 \\ 0 & I_s \end{bmatrix},\ \begin{bmatrix} \alpha & A' & 0 \\ 1 & 0 & I_s \end{bmatrix},\ \begin{bmatrix} \beta & 1 \\ A' & 0 \\ 0 & I_s \end{bmatrix},\ \begin{bmatrix} a & \beta & 1 \\ \alpha & A' & 0 \\ 1 & 0 & I_s \end{bmatrix}$$

respectively, i.e. replacing there

$$\mathrm{r}(A') \text{ with } \mathrm{r} \begin{bmatrix} A' & 0 \\ 0 & I_s \end{bmatrix} = 2^s\,\mathrm{r}(A'),$$

$$\mathrm{r} \begin{bmatrix} \alpha & A' \end{bmatrix} \text{ with } \mathrm{r} \begin{bmatrix} \alpha & A' & 0 \\ 1 & 0 & I_s \end{bmatrix} = (2^s - 1)\,\mathrm{r}(A') + \mathrm{r}(A_2),$$

$$\mathrm{r} \begin{bmatrix} \beta \\ A' \end{bmatrix} \text{ with } \mathrm{r} \begin{bmatrix} \beta & 1 \\ A' & 0 \\ 0 & I_s \end{bmatrix} = (2^s - 1)\,\mathrm{r}(A') + \mathrm{r}(A_3), \text{ and}$$

$$\mathrm{r} \begin{bmatrix} a & \beta \\ \alpha & A' \end{bmatrix} \text{ with } \mathrm{r} \begin{bmatrix} a & \beta & 1 \\ \alpha & A' & 0 \\ 1 & 0 & I_s \end{bmatrix} = (2^s - 1)\,\mathrm{r}(A') + \mathrm{r}(A_4)),$$

we obtain

$$\begin{aligned} \mathrm{r}(B') &= 2^{q+r+s}\,\mathrm{r}(A') + 2^{q+s}\,(\mathrm{r}(A_3) - \mathrm{r}(A')) + \\ &\quad + 2^{r+s}\,(\mathrm{r}(A_2) - \mathrm{r}(A')) + (\mathrm{r}(A_4) - \mathrm{r}(A_2) - \mathrm{r}(A_3)) \\ &= 2^{q+r+s}\,\mathrm{r}(A') + 2^{q+s} c_3 + 2^{r+s} c_2 + c_1. \end{aligned}$$

Multiplying this equality with 2^p and replacing $p+q+r+s$ with $n-k$, we obtain (16) for the remaining case $s > 0$. ☐

3. Characterization of Dense Matrices

Note that the classes $\mathcal{C}_i(A)$ have infinite number of elements. It turns out that the numbers of matrices in the sets $\mathcal{E}_i$, $i = 0, 1, 2$, of base matrices in $\mathcal{D}_i$, $i = 0, 1, 2$ are finite. Furthermore, we prove in Theorem 5 that for classes $\mathcal{F}_i$, $i = 0, 1, 2$, obtained by the following algorithm, holds that $\mathcal{E}_i = \mathcal{F}_i$.

Algorithm 2. Determine effectively the classes of matrices $\mathcal{F}_i$, $i = 0, 1, 2$.

Output : The classes $\mathcal{F}_i$, $i = 0, 1, 2$
begin
  // initialization
  set $n = 1$ and $\mathcal{G}_1 = \{I_1\}$
  while $|\mathcal{G}_n| > 0$ **do**
    set $\mathcal{G}_{n+1} = \emptyset$
    for all $A \in \mathcal{G}_n$ **do**
      if $\mathrm{type}(A) = i$, $i = 1, 2$ **then**
        insert $\mathrm{can}(\mathrm{Base}(A))$ into $\mathcal{F}_i$
      else if $\mathrm{type}(A) = 0$
        and A contains more than 3 1-blocks **then**
        insert $\mathrm{can}(\mathrm{Base}(A))$ into $\mathcal{F}_0$
      else
        for all $B \in \mathcal{B}_{n+1}$ such that
          $B(i,j) = A(i,j), 1 \le i, j \le n$ **do**
          if $\rho(B) > \frac{3}{8}$ **then**
            insert $\mathrm{can}(B)$ into $\mathcal{G}_{n+1}$
    replace n by $n + 1$
end

The maximum reached value of matrix dimension during the execution of the algorithm is $n = 12$, i.e. $\mathcal{G}_{13} = \emptyset$.

Definition 13. We say that A is **continue**-matrix if A is dense, $\mathrm{type}(A) = 0$, and A does not contain more that 3 1-blocks.

Theorem 4. *Let $n > 1$, and suppose $A \in \mathcal{B}_n$ is continue-matrix. Then there exists $C \in \mathcal{B}_{n-1}$, $C \subset A$, such that C is continue-matrix.*

Proof. If $n = 2$, from $\rho(A) > \frac{3}{8}$ it follows that $\mathrm{r}(A) > 1$, so there exist k and m, such that $A(k, m) = 1$, $k \ge 1$ and $m \le 2$. We can assume that $A(1, 1) = 1$. Then $C = A^{-2*,-*2}$ satisfies $\rho(C) = 1$, and C is continue-matrix.

Suppose now $n > 2$. If A contains a 1-block, then let C be matrix obtained from A by removing this 1-block. Then $\rho(C) = \rho(A) > \frac{3}{8}$, $\mathrm{type}(C) = 0$ and C contains at most two 1-blocks, hence C is continue-matrix.

Suppose now $n > 2$ and suppose A does not contain 1-block. Consider the following cases:

Case $\mathrm{rws}(A, n) = 0$:
: Since $\mathrm{type}(A) = 0$, A can contain at most one column of weight 4, so A contains a column with weight less than 4. We can assume that $\mathrm{w}(A(1,*)) = 0$ and $\mathrm{w}(A(*,1)) \leq 3$. Let $C = A^{-1*,-*1}$, then C contains at most 3 1-blocks, $\rho(C) \geq \rho(A)$ and $\mathrm{type}(C) = 0$, so C is continue-matrix.

Case $\mathrm{cws}(A, n) = 0$:
: The proof is analogous to the previous case.

Case $\mathrm{rws}(A, n) = 1$:
: We can assume that $A(1,1) = 1$ and $\mathrm{w}(A(1,*)) = 1$. Let $D = A^{-*1}$ and $C = A^{-1*,-*1}$. We have $\mathrm{r}(C) = \mathrm{r}(D) \geq \frac{\mathrm{r}(A)}{2}$ so $\rho(C) > \frac{3}{8}$. Let $E = A^{-1*}$. Furthermore, C contains at most 3 1-blocks, since $\mathrm{w}(E(1,*)) \leq 3$, E does not contain 1-block and $C = E^{-*1}$; hence C is continue-matrix.

Case $\mathrm{cws}(A, n) = 1$:
: The proof is analogous to the previous case.

Case $\mathrm{rws}(A, n), \mathrm{cws}(A, n) \geq 2$:
: From Theorem 1 it follows that there exists $C \in \mathcal{B}_{n-1}$ such that $C \subset A$ and $\rho(C) \geq \rho(A)$. We can assume $C = A^{-1*,-*1}$. From $\mathrm{type}(A) = 0$ it follows $\mathrm{type}(C) = 0$, hence C does not contain a row and column both of weight greater than 3. Since A does not contain row or column with weight less than two, the only 1-blocks in C could be for k, l such that $A(1,k) = 1$ and $A(m,1) = 1$. Since either $\mathrm{w}(A(1,*)) < 4$ or $\mathrm{w}(A(*,1)) < 4$, or both, C can not contain more than 3 1-blocks.

□

Lemma 20. *If A is an arbitrary continue-matrix, then $B = \mathrm{can}(\mathrm{Base}(A))$ is obtained by Algorithm 2.*

Proof. Let $\mathcal{F}_i, i = 1, 2, 3$ be the classes of matrices obtained by the Algorithm 2. Suppose $A \in \mathcal{B}_n$ for some $n \geq 1$. If $n = 1$, then $A = B = I_1 \in \mathcal{F}_0$, so suppose $n \geq 2$. From previous theorem it follows that there exists $C_{k-1} \subset A$, $C_{n-1} \in \mathcal{B}_{n-1}$ and C_{n-1} is continue-matrix. The same applies to C_{n-1}, so there exist C_i, $i = 1, 2, \ldots, n-1$, such that $I_1 = C_1 \subset C_2 \subset \cdots C_{n-1} \subset A$ and all C_i are continue-matrices, which means that A can be obtained with Algorithm 2. □

Note that this is a similar statement to Theorem 1. There are infinitely many dense matrices, but the classes $\mathcal{F}_i$, $i = 0, 1, 2$ are finite. This fact enables finding effectively all dense matrices, as the following theorem claims.

Theorem 5. *Let $\mathcal{F}_i$, $i = 1, 2, 3$ be the classes of matrices obtained by Algorithm 2. Then*

$$\mathcal{E}_i = \mathcal{F}_i, \quad i = 1, 2, 3$$

and

$$\mathcal{D} \subseteq \bigcup_{i=1,2,3} \bigcup_{A \in \mathcal{F}_i} \mathcal{C}_i(A). \tag{18}$$

Proof. From the fact that $\mathcal{F}_i$ are obtained by Algorithm 2, it follows that $\mathcal{F}_i \subseteq \mathcal{E}_i$ for every $1 \leq i \leq 3$. It remains to prove that $\mathcal{E}_i \subseteq \mathcal{F}_i$, i. e. that every dense base matrix is obtained by Algorithm 2. According to Lemma 20 it suffices to show for every matrix $A \in \mathcal{B}_n$, $A \in \mathcal{E}_i$, $0 \leq i \leq 2$, that A is continue-matrix or there exists a matrix $C \in \mathcal{B}_{n-1}$, $C \subset A$, such that C is continue-matrix. In either case A can be obtained by Algorithm 2. Consider the following:

Case $A \in \mathcal{E}_0$:
: Since A is dense, $\mathrm{type}(A) = 0$ and A doesn't contain 1-block, A is a continue-matrix. From Lemma 20 it follows that $A \in \mathcal{F}_0$.

Case $A \in \mathcal{E}_1$:
: Assume WLOG that $\mathrm{cws}(A, 1) > 4$ and $\mathrm{rws}(A, 1) \leq 3$ — otherwise, if $\mathrm{rws}(A, 1) > 4$ and $\mathrm{cws}(A, 1) \leq 3$, then consider A^T instead of A. Assume WLOG that $\mathrm{w}(A(*,1)) \geq 5$.

  Case $\mathrm{cws}(A, n) = 0$:
  : We can assume WLOG that $\mathrm{w}(A(*,2)) = 0$. Let $C = A^{-*1}$ and $D = A^{-*1-*2}$. Since $\mathrm{r}(C) = \mathrm{r}(D)$ and $D \in \mathcal{B}_{n,n-2}$ we have that $\mathrm{r}(C) \leq 2^{n-2}$ and $\mathrm{r}(A) \leq 2^{n-2} + 2^{n-5}$, hence A is not dense and $A \notin \mathcal{E}_1$, which contradicts the assumption.

  Case $\mathrm{cws}(A, n) = 1$:
  : Assume WLOG that $\mathrm{w}(A(*,2)) = 1$ and $A(1,2) = 1$. Since A is base matrix, it doesn't contain 1-block, so $\mathrm{w}(A(1,*)) > 1$. We now have two possibilities:

    1) $\mathrm{w}(A(1,*)) = 2$. Again, we have two cases:

       a) If $A(1,1) = 1$ then A contains a cross $\mathcal{K}_{1,0}(A, *, 1, 1, 2)$ and since A is base matrix, there has to be $A(*,1) = 5$. Let $D = A^{-1*}$ and $C = A^{-1*,-*2}$. Since $\mathrm{r}(C) = \mathrm{r}(D)$ we have that $2\,\mathrm{r}(C) \geq \mathrm{r}(A)$, hence C is dense. Since $\mathrm{rws}(C, 1) < 4$ and $\mathrm{cws}(C, 1) < 4$, and C can not contain 1-block, C is continue-matrix.

       b) If $A(1,1) = 0$, let $D = A^{-*1}$ and $C = A^{-1*,-*1}$. From Lemma 10 it follows that $\rho(C) \geq \rho(A)$. If C contains less than 4 1-blocks, then C is a continue-matrix, and we are done. Otherwise, D has to contain at least two 1-blocks, i.e. for some $1 < i \leq n$ and $2 < j \leq n$, A contains a cross $\mathcal{K}_{1,0}(A, *, 1, i, j)$. Since A is base matrix, $\mathrm{w}(A(*,1)) = 5$. Then, let $E = A^{-i*,-*j}$. According to Lemma 13 E is dense matrix, and since $A(i,1) = 1$ we have $\mathrm{cws}(E, 1) = 4$. We also have that $\mathrm{rws}(E, 1) \leq 3$, and since

E doesn't contain 1-blocks, E is continue-matrix, so we are done.

2) $\mathrm{w}(A(1,*) = 3$. Let $C = A^{-1*,-*1}$. According to Lemma 10 $\rho(C) \geq \rho(A)$, i. e. C is dense. We have that $\mathrm{cws}(C) < 4$ and $\mathrm{rws}(C) < 4$, and if C contains less than four 1-blocks C is continue-matrix, and we are done. Otherwise, A has to contain a cross $\mathcal{K}_{1,0}(A,*,1,i,j)$. But then for $E = A^{-i*,-*j}$ and according to Lemma 13 E is dense matrix, and since $A(i,1) = 1$ we have $\mathrm{cws}(E,1) = 4$. We also have that $\mathrm{rws}(E,1) \leq 3$, and since E doesn't contain 1-blocks, E is continue-matrix, so we are done.

Case $\mathrm{cws}(A,n) \geq 2$:

The sum of row weights equals to the sum of column weights, which is more than $2n$, implying that A contains a row with weight larger than 2. Since $\mathrm{rws}(A,1) \leq 3$ it follows $\mathrm{rws}(A,1) = 3$. Assume WLOG that $\mathrm{w}(A(1,*)) = 3$. Let $C = A^{-1*,-*1}$. From Lemma 10 it follows that C is dense. Since A doesn't contain 1-block, and since $\mathrm{w}(A(*,i)) \geq 2$ for every $1 \geq i \geq n$, C has at most three 1-blocks, hence C is a continue-matrix.

Case $A \in \mathcal{E}_2$:

Consider the following cases:

Case $\mathrm{rws}(A,1), \mathrm{cws}(A,1) \geq 5$:

Assume WLOG that $\mathrm{w}(A(1,*)), \mathrm{w}(A(*,1)) > 4$. Let $C = A^{-1*,-*1}$. Since A is base matrix, it can not contain a cross $\mathcal{K}_{1,0}(A,1,1,k,l)$ or $\mathcal{K}_{0,1}(A,1,1,k,l)$, nor it can contain two crosses $\mathcal{K}_{1,1}(A,1,1,k_1,l_1)$, $\mathcal{K}_{1,1}(A,1,1,k_2,l_2)$ such that $k_1,l_1,k_2,l_2 > 1$, $k_1 \neq l_1, k_2 \neq l_2$. Hence, C contains at most one 1-block. Furthermore, $\mathrm{type}(C) = 0$ because $\mathrm{rws}(C,1), \mathrm{cws}(C,1) \leq 3$, and C is dense by Lemma 10. Therefore C is a continue-matrix.

Case $\mathrm{cws}(A,1) \geq 5$, $\mathrm{rws}(A,1) = 4$:

Assume WLOG that $\mathrm{w}(A(1,*)) = 4$ and $\mathrm{w}(A(*,1)) > 4$. Since A is a base matrix, it does not contain a cross $\mathcal{K}_{1,0}(A,1,1,k,l)$. Let $C = A^{-1*,-*1}$. By Lemma 10, C is dense. Now, we have two possibilities:

1) C contains at most three 1-blocks. Since $\mathrm{rws}(C,1) \leq 3$ and $\mathrm{cws}(C,1) \leq 3$ we have that $\mathrm{type}(C) = 0$, and therefore C is a continue-matrix.
2) C contains four 1-blocks. Again, we consider cases:
   a) $\mathrm{rws}(C,1) = 1$. If $\mathrm{rws}(C,n-1) = 0$ then WLOG we assume that $\mathrm{w}(C(1,*)) = 0$, hence $\mathrm{r}(A^{-*1}) = \mathrm{r}(A^{-2*,-*1})$and $A^{-2*,-*1}$ is continue-matrix.
   So, we consider now the case when $\mathrm{rws}(C,n-1) > 0$, i.e. $\mathrm{rws}(C,n-1) = 1$. If $\mathrm{cws}(C,n-1) = 0$ and since for all columns i such that $A(1,i) = 1$ is $\mathrm{w}(A(*,i)) = 2$ (since C contains four 1-blocks and $\mathrm{w}(A(1,*)) = 4$), hence $\mathrm{cws}(A,n) = 0$. We can assume WLOG that $\mathrm{w}(A(*,2)) = 0$, so $\mathrm{r}(A) \leq 2^{n-2} + 2^{n-5}$ so A is not dense, which contradicts the assumption. If $\mathrm{rws}(C,n-1) > 0$, since $\mathrm{w}(C(i,*)) = 1$ for every $1 \leq i \leq n-1$, $C = I_{n-1}$, which implies that A is not base matrix, which contradicts the assumption.
   b) $\mathrm{rws}(C,1) \geq 2$. We can assume that $\mathrm{w}(C(1,*)) \geq 2$. Let $D = A^{-2*,-*1}$. Since $\mathrm{type}(D) = 0$, D is dense according to Lemma 10, D contains at most three 1-blocks, hence D is continue-matrix.

Case $\mathrm{rws}(A,1) \geq 5$, $\mathrm{cws}(A,1) = 4$:

The case is analogous to previous case.

Case $\mathrm{rws}(A,1) = \mathrm{cws}(A,1) = 4$:

We can assume that $\mathrm{w}(A(1,*)) = 4$ and $\mathrm{w}(A(*,1)) = 4$. Let $C = A^{-1*,-*1}$. Since $\mathrm{type}(C) = 0$, C is dense (Lemma 10), and if it contains less then four 1-blocks C is a continue-matrix, and we are done. Suppose that C has four 1-blocks. C contains 1-blocks only in the following cases:

1) A contains a cross $\mathcal{K}_{1,0}(A,1,1,k,l)$. Then $D = A^{-k*,-*l}$ is dense, $\mathrm{type}(D) = 0$ and D does not contain 1-block, so D is a continue-matrix.
2) A contains a cross $\mathcal{K}_{0,1}(A,1,1,k,l)$. Then $D = A^{-k*,-*l}$ is a continue-matrix, as in the previous case.
3) A contains two crosses $\mathcal{K}_{1,1}(A,1,1,k_1,l_1)$, $\mathcal{K}_{1,1}(A,1,1,k_2,l_2)$ such that $k_1,l_1,k_2,l_2 > 1$, $k_1 \neq l_1, k_2 \neq l_2$. Let $D = A^{-k_1*,-*l_1}$. From Lemma 18 it follows that $\rho(D) \geq \rho(A)$. Furthermore, $\mathrm{type}(D) = 0$, since $A(1,l_1) = 1$ and $A(k_1,1) = 1$, and D does not contain 1-blocks, hence D is a continue-matrix.

The statement (18) follows from $\mathcal{E}_i = \mathcal{F}_i$, Lemma 19, and Algorithm 1. □

TABLE I
THE SET OF TRIPLETS $(k, \mathrm{r}(A), \mathrm{w})$ FOR ALL THE MATRICES A IN THE SHORTENED LIST $\mathcal{E}_0$.

	k	$\mathrm{r}(A)$	w		k	$\mathrm{r}(A)$	w
1	1	1	1	26	7	50	3
2	1	2	1	27	7	51	4
3	2	2	1	28	7	52	3
4	2	3	2	29	7	53	4
5	3	4	1	30	7	54	4
6	3	5	2	31	7	56	3
7	4	7	3	32	7	57	4
8	4	8	1	33	7	58	4
9	4	9	2	34	7	64	1
10	4	10	2	35	7	65	2
11	5	13	3	36	8	97	3
12	5	14	3	37	8	98	3
13	5	15	4	38	8	99	4
14	5	16	1	39	8	100	3
15	5	17	2	40	8	101	4
16	6	25	3	41	8	102	4
17	6	26	3	42	8	104	3
18	6	27	4	43	8	105	4
19	6	28	3	44	8	106	4
20	6	29	4	45	8	108	4
21	6	30	4	46	8	112	3
22	6	31	5	47	8	128	1
23	6	32	1	48	9	195	4
24	6	33	2	49	9	200	3
25	7	49	3	50	9	208	3

4. THE MAIN THEOREM

By executing Algorithm 2, the lists $\mathcal{E}_i$, $i = 0, 1, 2$ are obtained. They are shortened by further simplifying the elements $A \in \mathcal{E}_i$, $i = 1, 2$: while $A \in \mathcal{B}_k$, $k > 1$, has a cross $\mathcal{K}_{x,y}(A, 1, 1, k, l)$, $x+y > 0$, the row k and the column l are removed from A. Let $\mathrm{w}(m)$ denote the number of ones in the binary form of m. In Tables I, II, and III, the following values for the matrices $A \in \mathcal{B}_k$ in shortened lists $\mathcal{E}_i$, are presented:

Table I:
the values k, $\mathrm{r}(A)$ and $\mathrm{w} = \mathrm{w}(\mathrm{r}(A))$.

Table II:
for $A = \begin{bmatrix} \alpha & A' \end{bmatrix}$ the values: k, $c_2 = \mathrm{r}(A) - \mathrm{r}(A')$, and $\mathrm{w} = \mathrm{w}(\mathrm{r}(A')) + \mathrm{w}(c_2)$ (see Theorem 2).

Table III:
for $A = \begin{bmatrix} a & \beta \\ \alpha & A' \end{bmatrix}$ the values: k, $\mathrm{r}(A')$, c_2, c_3, c_1 and $\mathrm{w} = \mathrm{w}(\mathrm{r}(A')) + \mathrm{w}(c_2) + \mathrm{w}(c_3) + \mathrm{w}(c_1)$ (see Theorem 3). Each row of Table III also contains an example matrix, written as an array of integers whose binary representations are the matrix rows.

The last column in each of these tables is the upper bound on $\mathrm{w}(\mathrm{r}(A))$ for corresponding $A \in \mathcal{E}_i$, for:

$i = 0$:
$\mathrm{r}(\mathcal{X}_{0,0}^{p}(A))$, $p + k = n$

$i = 1$:
$\mathrm{r}(\mathcal{X}_{0,0}^{p}\mathcal{X}_{1,0}^{q}(A))$, $p + q + k = n$

$i = 2$:
$\mathrm{r}(\mathcal{X}_{0,0}^{p}\mathcal{X}_{1,0}^{q}\mathcal{X}_{0,1}^{r}\mathcal{X}_{1,1}^{s}(A))$, $p + q + r + s + k = n$

Using these results, we now prove that the conjecture $\mathcal{R}_n \cap (2^{n-2} + 2^{n-3}, 2^{n-1}) = \mathcal{A}_n$ is in fact true.

TABLE II
THE SET OF QUADRUPLES $(k, \mathrm{r}(A'), c_2, \mathrm{w})$ FOR ALL THE MATRICES A IN THE SHORTENED LIST $\mathcal{E}_1$.

	k	$\mathrm{r}(A')$	c_2	w
1	1	1	1	2
2	1	1	0	1
3	2	2	1	2
4	2	2	0	1
5	3	3	3	4
6	3	4	1	2
7	3	3	2	3
8	3	4	0	1
9	3	3	1	3
10	4	7	3	5
11	4	8	1	2
12	4	7	2	4
13	4	6	3	4
14	4	8	0	1
15	4	7	1	4
16	4	6	2	3
17	4	7	0	3
18	4	6	1	3
19	5	13	4	4
20	5	12	5	4
21	5	13	3	5
22	5	12	4	3
23	5	13	2	4
24	5	12	3	4
25	5	13	1	4
26	5	12	2	3
27	5	13	0	3
28	5	12	1	3
29	6	24	8	3
30	6	24	6	4
31	6	24	4	3
32	6	24	3	4
33	6	24	2	3
34	6	23	3	6
35	6	24	1	3

Theorem 6. *Let $\mathcal{A}_n = \mathcal{A}_{n,3} \cup \mathcal{A}_{n,4} \cup \mathcal{A}'_{n,5} \cup \mathcal{A}''_{n,5}$, see* (1). *Then*

$$\mathcal{R}_n \cap (2^{n-2} + 2^{n-3}, 2^{n-1}) = \mathcal{A}_n.$$

Proof. We prove first that if $a \in \mathcal{A}_n$, then $a = \mathrm{r}(A)$ for some $A \in \mathcal{B}_n$. Note that this is in fact proved by another construction in [9]. Let $A = \begin{bmatrix} a & \beta \\ \alpha & A' \end{bmatrix}$.

Case $a \in \mathcal{A}_{n,3}$:

Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \in \mathcal{E}_2 \cap \mathcal{B}_3.$$

from row 13 of Table III. We have $\mathrm{r}(A') = 3$, $c_2 = c_3 = 0$, $c_1 = 1$ (the case $s = 0$ in (17), Theorem 3), hence i may be chosen, $0 \le i \le n-4$, such that

$$\mathrm{r}\left(\mathcal{X}_{0,1}^{n-3-i}\mathcal{X}_{0,0}^{i}(A)\right) = 2^{n-2} + 2^{n-3} + 2^{i} = a,$$

i.e. $\mathcal{R}(\mathcal{C}_2(A) \cap \mathcal{B}_n) \supseteq \mathcal{A}_3$.

Case $a \in \mathcal{A}_{n,4}$:

Consider next the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \in \mathcal{E}_2 \cap \mathcal{B}_3.$$

TABLE III
THE SET OF QUINTUPLES $(k, \mathrm{r}(A'), c_2, c_3, c_1, \mathrm{w})$ FOR ALL THE MATRICES A IN THE SHORTENED LIST $\mathcal{E}_3$.

	the matrix A					k	$\mathrm{r}(A')$	c_2	c_3	c_1: $s>0$	c_1: $s=0$	w: $s>0$	w: $s=0$
1	1					1	1	0	0	0	1	1	2
2	0					1	1	0	0	1	0	2	1
3	4	1	3			3	3	0	0	0	3	2	4
4	1	3	6			3	3	1	1	2	0	5	4
5	5	6	3			3	3	1	1	0	0	4	4
6	2	1	7			3	3	0	1	2	1	4	3
7	4	5	3			3	3	0	1	0	1	3	4
8	4	1	7			3	3	0	0	0	2	2	3
9	0	5	3			3	3	0	1	2	0	4	3
10	1	6	7			3	3	0	1	1	0	4	3
11	6	1	7			3	3	0	1	0	0	3	3
12	1	5	7			3	3	0	0	1	1	3	3
13	4	5	7			3	3	0	0	0	1	2	3
14	0	1	3			3	3	0	0	3	0	4	2
15	0	1	7			3	3	0	0	2	0	3	2
16	0	6	7			3	3	0	0	1	0	3	2
17	10	9	5	6		4	5	2	2	0	1	4	5
18	12	9	10	7		4	5	1	3	0	0	5	5

from row 7 of Table III. We have $\mathrm{r}(A') = 3$, $c_2 = 0$, $c_3 = c_1 = 1$ (the case $s = 0$ in (17), Theorem 3), hence i, j may be chosen, such that $0 \le j < i \le n-3$, and

$$\mathrm{r}\left(\mathcal{X}_{1,0}^{n-3-i}\mathcal{X}_{0,1}^{i-j}\mathcal{X}_{0,0}^{j}(A)\right) = 2^{n-2} + 2^{n-3} + 2^i + 2^j = a,$$

i.e. $\mathcal{R}(\mathcal{C}_2(A) \cap \mathcal{B}_n) \supseteq \mathcal{A}_4$.

Case $a \in \mathcal{A}'_{n,5}$:

Consider the matrix

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \in \mathcal{E}_2 \cap \mathcal{B}_3.$$

From row 4 of Table III we can see that $\mathrm{r}(A') = 3$, $c_2 = c_3 = 1$ and $c_1 = 2$ (corresponding the case $s > 0$ in (17), Theorem 3), hence i, j, k may be chosen, such that $0 \le k$, $k+1 < i \le n-4$, and

$$\mathrm{r}(\mathcal{X}_{1,1}\mathcal{X}_{1,0}^{i-k}\mathcal{X}_{0,0}^{k}(A)) = 2^{n-2} + 2^{n-3} + 2^i + 2^{k+1} + 2^k = a$$

i.e. $\mathcal{R}(\mathcal{C}_2(A) \cap \mathcal{B}_n) \supseteq \mathcal{A}'_{n,5}$.

Case $a \in \mathcal{A}''_{n,5}$:

Using the same matrix A from the row 4 of Table III, we may choose i, j, k, such that $1 \le k$, $k+2 \le j < i$, $i + j \le n + k - 5$, and

$$\mathrm{r}(\mathcal{X}_{1,1}^{s}\mathcal{X}_{1,0}^{j+1}\mathcal{X}_{0,1}^{i+1}\mathcal{X}_{0,0}^{k-1}(A)) = 2^{n-2} + 2^{n-3} + 2^{k+i} + 2^{k+j} + 2^k = a,$$

i.e. $\mathcal{R}(\mathcal{C}_2(A) \cap \mathcal{B}_n) \supseteq \mathcal{A}''_{n,5}$.

Hence, $\mathcal{R}'_n \supseteq \mathcal{A}_n$.

Now we prove

$$\mathcal{R}'_n = \{\mathrm{r}(B) \mid B \in \bigcup_{i=1,2,3}\bigcup_{A \in \mathcal{E}_i} \mathcal{C}_i(A) \cap \mathcal{B}_n\} \cap (2^{n-2} + 2^{n-3}, 2^{n-1}) \subseteq \mathcal{A}_n.$$

Note that

$$\mathcal{A}_{n,i} = \{a \in (2^{n-2} + 2^{n-3}, 2^{n-1}) \mid \mathrm{w}(a) = i\}, \quad i = 3, 4$$

Note also that

$$\mathcal{A}'_{n,5} = \{a \in (2^{n-2} + 2^{n-3}, 2^{n-1}) \mid \mathrm{w}(a) = 5 \tag{19}$$
$$\text{and } a = 2^{n-2} + 2^{n-3} + 2^i + 2^{k+1} + 2^k, \tag{20}$$
$$0 \le k,\ k+2 \le i \le n-4\}$$

contains all integers in $(2^{n-2}+2^{n-3}, 2^{n-1})$ with five binary ones, such that their lowest two binary ones are adjacent. Hence we need to check only those matrices A that can be obtained from Tables I, II, and III, such that $\mathrm{w}(\mathrm{r}(A)) \ge 5$, and if $\mathrm{w}(\mathrm{r}(A)) = 5$, then the two lowest binary ones of $\mathrm{w}(\mathrm{r}(A))$ are not adjacent.

Table I:

Only the entry 22 has $\mathrm{w} \ge 5$. If $A \in \mathcal{B}_6$ is such that $\mathrm{r}(A) = 31$, and $B = \mathcal{X}_{0,0}^{n-6}(A)$, then $\mathrm{r}(B) = 31 \cdot 2^{n-6} \in \mathcal{A}'_{n,5}$ (because the lowest two binary ones of $\mathrm{r}(B)$ are adjacent).

Table II:

Let $A = [\alpha\ A'] \in \mathcal{B}_k$, be the matrix corresponding to some entry in Table II, where $A' \in \mathcal{B}_{k,k-1}$ and $\alpha \in \mathcal{B}_{k,1}$. Let

$$B = \mathcal{X}_{0,0}^{p}\mathcal{X}_{1,0}^{q}(A) \in \mathcal{B}_n \cap \mathcal{C}_1(A), \quad p + q + k = n.$$

Then $\mathrm{r}(B) = 2^{n-k}\,\mathrm{r}(A') + 2^p c_2$, see Theorem 2. In Table II there are two entries with $\mathrm{w} = 5$ (10, 21) and one with $\mathrm{w} = 6$ (34). The entries 10 and 21 have $c_2 = 3$, implying that if $\mathrm{w}(\mathrm{r}(B)) = 5$, then $\mathrm{r}(B) \in \mathcal{A}'_{n,5}$ (the two lowest binary ones are adjacent).

Let B be the matrix obtained from the matrix corresponding to entry 34 in Table II. If B is dense, then $\mathrm{r}(B) = 23 \cdot 2^{n-6} + 3 \cdot 2^p > 2^n \cdot 3/8$, which is equivalent to $p \ge n-7$. But from $p + q = n-6$ it follows $p \le n-6$, hence $p \in \{n-7, n-6\}$.

Consequently, $\mathrm{r}(B) \in \{49 \cdot 2^{n-7}, 13 \cdot 2^{n-5}\}$, $\mathrm{w}(\mathrm{r}(B)) = 3$ and $\mathrm{r}(B) \in \mathcal{A}_{n,3}$ in both these cases.

Table III:

Let $A = \begin{bmatrix} a & \beta \\ \alpha & A' \end{bmatrix}$ be the matrix corresponding to some entry in Table III, where $A' \in \mathcal{B}_{k-1,k-1}$. Let

$$B = \mathcal{X}_{0,0}^p \mathcal{X}_{1,0}^q \mathcal{X}_{0,1}^r \mathcal{X}_{1,1}^s(A), \quad p+q+r+s+k = n.$$

Then from Theorem 3 it follows

$$\mathrm{r}(B) = 2^{n-k}\,\mathrm{r}(A') + 2^{p+q}c_2 + 2^{p+r}c_3 + 2^p c_1, \quad p+q+r+s = n.$$

There are three entries (4 with $s > 0$, 17 with $s = 0$, and 18 with $s \geq 0$) with $\mathrm{w} \geq 5$.

entry 4 with $s > 0$:

We have already considered the entry 4 in the first part of the proof, showing that it covers $A'_{n,5}$ and $A''_{n,5}$. If A is the corresponding matrix, then

$$\mathrm{r}(B) = 3 \cdot 2^{n-3} + 2^{p+q} + 2^{p+r} + 2^{p+1}, \quad p+q+r < n-3.$$

Obviously, $\mathrm{r}(B) > 3/8 \cdot 2^n$, so B is always dense. If $q = r$, then $\mathrm{r}(B) = 3 \cdot 2^{n-3} + 2^{p+q+1} + 2^{p+1}$, implying $\mathrm{w}(\mathrm{r}(B)) \leq 4$ and $\mathrm{r}(B) \in \mathcal{A}_n$. Suppose WLOG $r > q$. If $q = 0$, then

$$\mathrm{r}(B) = 3 \cdot 2^{n-3} + 2^{p+r} + 2^{p+1} + 2^p, \quad p+r < n-3,$$

i.e. $\mathrm{r}(B) \in \mathcal{A}'_{n,5}$. If $q = 1$, then $\mathrm{w}(\mathrm{r}(B)) \leq 4$. If $q = 2$ and $\mathrm{w}(\mathrm{r}(B)) = 5$, then $\mathrm{r}(B) \in \mathcal{A}'_{n,5}$. Otherwise if $q \geq 2$, then $p+q+r < n-3$ is equivalent to $(p+r)+(p+q) \leq n-5+(p+1)$, and we have $\mathrm{r}(B) \in \mathcal{A}''_{n,5}$.

entry 17 with $s = 0$:

If A is the corresponding matrix, then

$$\mathrm{r}(B) = 5 \cdot 2^{n-4} + 2^{p+q+1} + 2^{p+r+1} + 2^p, \quad p+q+r = n-4.$$

If $q = r$, then $\mathrm{w}(\mathrm{r}(B)) \leq 4$, hence suppose WLOG $q > r$. The condition $\mathrm{r}(B) > 3/8 \cdot 2^n$ is equivalent to $2^{p+q+1} + 2^{p+r+1} + 2^p > 2^{n-4}$, i.e. $p+q \geq n-5$. From the other side, $p+q = n-4-r \leq n-4$, hence $p+q \in \{n-5, n-4\}$. If $p+q = n-5$, then $r = 1$, $\mathrm{r}(B) = 3 \cdot 2^{n-3} + 2^{p+2} + 2^p$, and $\mathrm{w}(\mathrm{r}(B)) \leq 4$. Otherwise, if $p+q = n-4$, then $r = 0$ and $\mathrm{r}(B) = 3 \cdot 2^{n-3} + 2^{n-4} + 2^{p+1} + 2^p$, implying $\mathrm{r}(B) \in \mathcal{A}'_{n,5}$.

entry 18 with $s \geq 0$:

If A is the corresponding matrix, then

$$\mathrm{r}(B) = 5 \cdot 2^{n-4} + 3 \cdot 2^{p+q} + 2^{p+r},$$

where $p+q+r \leq n-4$.

Case $r = q$:

$\mathrm{r}(B) = 5 \cdot 2^{n-4} + 2^{p+q+2}$, hence $\mathrm{w}(\mathrm{r}(B)) \leq 3$.

Case $r = q-1$:

$\mathrm{r}(B) = 5 \cdot 2^{n-4} + 2^{p+q+1} + 2^{p+q-1}$, hence $\mathrm{w}(\mathrm{r}(B)) \leq 4$.

Case $r < q-1$:

$\mathrm{r}(B) > 3/8 \cdot 2^n$ is equivalent to $p+q \geq n-5$. From $p+q+r \leq n-4$ it follows $p+q \leq n-4$, hence $p+q \in \{n-4, n-5\}$. If $p+q = n-4$, then $\mathrm{r}(B) = 2^{n-1} + 2^p > 2^{n-1}$. Otherwise if $p+q = n-5$, then $\mathrm{r}(B) = 13 \cdot 2^{n-5} + 2^{p+r}$ and $\mathrm{w}(\mathrm{r}(B)) \leq 4$.

Case $r > q+1$:

$\mathrm{r}(B) > 3/8 \cdot 2^n$ is equivalent to $p+r \geq n-4$. From $p+q+r \leq n-4$ it follows $p+r = n-4$ and $q = 0$, hence $\mathrm{r}(B) = 3 \cdot 2^{n-3} + 3 \cdot 2^p$ and $\mathrm{w}(\mathrm{r}(B)) \leq 4$.

□

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